Lecture 19: Online Algorithms & Paging

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Overview

- Part I
  - Why Study Online Algorithms?
  - Competitive Analysis
  - Examples
- Paging & Caching
- Conclusion
- Acknowledgements

Why Study Online Algorithms?

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.

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- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...

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- Applications in
  - Stock Market
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  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future* (that is, see the entire input beforehand)[1]

  "offline algorithm"

  1. Worst-case analysis

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    2. worst-case analysis
- Today, we will only see algorithms which must deal with the input as it receives it, *no constraints in memory*.
    1. Goal here is to *be competitive* against *any offline algorithm* (that is, algorithms that could see the entire input beforehand)
    2. worst-case analysis

Competitive Analysis

- Input is given as a sequence $s = s_1, s_2, \ldots, s_n$ of events.

Assume that there is some cost that we are trying to minimize.

Given algorithm A

$C_A(s)$:= cost of algorithm A on input s

Competitive Analysis

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Definition (Deterministic Competitive Ratio)

A deterministic online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

$$C_A(s) \leq k \cdot C_{opt}(s) + O(1)$$

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Definition (Randomized Competitive Ratio)

A randomized online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

$$\mathbb{E}[C_A(s)] \leq k \cdot C_{opt}(s).$$

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  3. If we go 10 times, it doesn't matter which way it goes...

If we knew exactly how many times we would go skiing in our lifetime then could act optimally

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- How is this an online algorithm?

sequence : days of our lives

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- Each time we go skiing, we have to decide whether to buy or rent (unless we bought it beforehand)

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- How is this an online algorithm?
- Each time we go skiing, we have to decide whether to buy or rent (unless we bought it beforehand)
- Algorithm has to decide *when to buy*, knowing only that we have gone skiing $\underbrace{t \text{ times}}_{t-1 \text{ times}}$

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  - If $t \leq 9$, then rent
  - When $t = 10$, buy

t = (# times we went skiing in the past) + 1

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- Analysis:
  - If $t \leq 9$, then best strategy is to rent: so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot t}{100 \cdot t} = 1$$

OPT: just rent $\quad C_{opt} = 100 \cdot t$

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  - If $t \geq 10$, we buy at the 10^{th} time, so cost is:

$$\frac{C_A}{C_{opt}} = \frac{\boxed{100 \cdot 9} + 1000}{1000} = \boxed{1.9}$$

$t \geq 10$ OPT: buy in beginning

C_A = cost of renting 9 times + cost of buy

~~Secretary~~ Dating Problem

- In the high-tech life, you decide to join a dating site...

[3] Assumptions: people are comparable AND we know how to do it
[4] Go big or go ~~home~~ lonely!
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- **Goal:** maximize probability of dating the best person

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- Consider the following algorithm:

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- Consider the following algorithm:
    1. Let's assume that all people you want to date are ranked and associate them with their rank: $1, 2, \ldots, n$

- we don't know this rank in advance

- but we DO KNOW how to compare ranks of two people we have seen

(relative rank)

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- More general algorithm: given a time t, go on t dates and from date $t+1$ onwards you decide to settle with a person who is better than the previous ones.

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- Suppose we pick a person at time k, then want to compute probability

$$P_k = \Pr[\pi(k) = 1 \text{ and we pick person at time } k]$$

reject

$\pi(1) = 2$ $\quad \pi(2) = 10 \quad \pi(3) = 27$

$\cancel{\pi(4) = 42} \quad \cancel{\pi(5) = 3}$ $\quad \boxed{\pi(6) = 1}$

did not pick

$t = 3$
$k = 6$

π	1	2	3	4	5	6	...
rank	2	10	27	42	3	1	

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- If $\pi(k) = 1$, then $1 - P_k$ is the probability that we picked a person between $[t+1, k-1]$, which means someone in this range better than the first t people. (and the people before it)

$$P_k = \Pr[\pi(k) = 1 \text{ and } \underbrace{\min \pi(1), \ldots, \pi(k-1) \text{ is in } \{\pi(1), \ldots, \pi(t)\}}_{\text{we did not pic anyone in } [t+1, k-1]}]$$

$\Leftrightarrow$ min rank before k appears in the first t

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$$P_k = \Pr[\pi(k) = 1 \text{ and } \underbrace{\min \pi(1), \ldots, \pi(k-1) \text{ is in } \{\pi(1), \ldots, \pi(t)\}}_{\text{min of first k-1 dates is in first t dates}}]$$

$$= \boxed{\frac{1}{n}} \cdot \boxed{\frac{t}{k-1}}$$

$\Pr[\pi(k) = 1]$

$\Pr[\pi_1$ falls in first t positions (out of $k-1$) of random permutation$]$ $= \frac{t}{k-1}$

conditioned on $\pi(k) = 1$

$\underbrace{\pi_1, \pi_2, \ldots, \pi_{k-1}}_{\text{values that we picked in first k-1 dates}} \in \{2, 3, \ldots, n-1\}$

$\boxed{\pi_1} < \pi_2 < \pi_3 < \cdots < \pi_{k-1}$

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- We get

$$P = \sum_{k>t}^{n} \frac{1}{n} \cdot \frac{t}{k-1} = \frac{t}{n} \cdot \sum_{k>t}^{n} \frac{1}{k-1} \approx \frac{t}{n} \cdot (\ln n - \ln t) = \frac{t}{n} \cdot \ln(n/t)$$

$$\sum_{a=1}^{n} \frac{1}{a} - \sum_{a=1}^{t} \frac{1}{a}$$

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- Optimizing we get that we should set $t = n/e$, which gives us $1/e$ probability.

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- Optimizing we get that we should set $t = n/e$, which gives us $1/e$ probability.
- Wait a second, where is the competitive analysis?

Making Dating Competitive

- To make the dating problem competitive, we would have to modify it a little bit.
  - We can simply minimize the rank.

New goal

Making Dating Competitive

- To make the dating problem competitive, we would have to modify it a little bit.
  - We can simply minimize the rank.
  - Say we always want to end up with someone (loneliness has a cost of $-\infty$, after all nobody wants to be alone)

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we know $\boxed{C_{OPT} = 1}$

$\therefore$ our algorithm will be $\Theta(n)$-competitive

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- That is, as we get older, we become more desperate to find someone and lower our expectations...

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- Computer memory is hierarchical: cache $\rightarrow$ L1 $\rightarrow$ L2 $\rightarrow$ main memory

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How to delete from cache?

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- Cost function: *number of cache misses*
- Simplification: assume we only have cache and main memory.

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Today, we will analyze the **Least Recently Used** heuristic. We will assume that *the size of our cache is k pages*.

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Today, we will analyze the **Least Recently Used** heuristic. We will assume that *the size of our cache is k pages*.

1. **Least Recently Used (LRU)**: k-competitive
2. **Random**: k-competitive
3. **First-in, First-out (FIFO)**: k-competitive
4. **Least Frequently Used (LFU)**: NOT competitive

LRU Analysis

Theorem

For cache of size k, LRU is k-competitive.

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For cache of size k, LRU is k-competitive.

1. Upper bound: divide input sequence into phases.
   - First phase starts immediately after our algorithm first faults, ends right after the algorithm faults k more times
   - Second phase starts at end of first phase, ends when algorithm faults for additional k times
   - and so on...

algorithm faults $\leftrightarrow$ cache miss

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   any algorithm must
2. We will prove that OPT algorithm faults *at least* once per phase

per phase LRU faults k times (by definition of phase)

OPT faults ≥ 1 time (we'll prove)

$\Rightarrow C_{LRU} \leq k \cdot C_{OPT}$

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4. Examples of phases, for $k = 3$:

   1, 1, 2, 2, 1, 3, 4, 3, 2, 4, 5, 6, 15, 4, 4, 2, 3, 5, 6, 4,5

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1 (~~1~~, $\boxed{2}$, ~~2~~, ~~1~~, $\boxed{3}$, $\boxed{4}$) (3, 2, 4, 5, 6) (15, 4, 4, 2) (3, 5, 6) (4, 5

3 faults

LRU Analysis - Example

Examples of phases, for $k = 3$:

$$\boxed{1}\ (\cancel{1}, \boxed{2}, \cancel{2}, \cancel{1}, \boxed{3}, \boxed{4})\ (\cancel{3}, \boxed{2}, \cancel{4}, \boxed{5}, \boxed{6})\ (15, 4, 4, 2)\ (3, 5, 6)\ (4, 5$$

phase 1

[ø | ø | ø] → [1 | ø | ø] → [1 | 2 | ø] → [1 | 2 | 3] →

→ [1 | 4 | 3] → [2 | 4 | 3] → [2 | 4 | 5] → [6 | 4 | 5]

and so on ..

LRU Analysis - Upper Bound

- Need to prove that OPT will fault at least once per phase.
- If the same page faulted twice in one phase:

had no p

r_1 $\boxed{r_2 \ \ldots}$ r_m

page p

page p

within a phase

must have queried k-1 (different) other pages

$\Rightarrow$ must have made k+1 distinct requests in this phase

$\Rightarrow$ Pigeonhole principle tells us that OPT must have faulted (OPT has memory $\leq k$)

page p was not in cache

in green period p was removed from cache (at r_m request p and get page fault)

using LRU heuristic p at some point was furthest in past $\Leftrightarrow$ requested all other k-1 pages after last request for p

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- If each page faulted once in a phase.

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- **Claim:** in the beginning of each phase, content of OPT and content of our algorithm A intersect in at least one page.
- Proof: Look at last fault page in previous phase.

.... b)

last fault of previous phase then we will have b in our cache in current phase

1) if OPT faulted at b then OPT will have b in its cache

2) if OPT did not fault, then b must have been in OPT's cache (only way not to fault)

LRU Analysis - Upper Bound

- If each page faulted once in a phase.
- **Claim:** in the beginning of each phase, content of OPT and content of our algorithm A intersect in at least one page.

- Since OPT and A had a common page, then OPT must have faulted as well (since each page faulted in this phase)

in particular b also faulted.

Concludes our proof of $\leq k$-competitive

Lower Bound - Deterministic Paging Algorithms

Theorem

Any deterministic algorithm for paging with k pages is at least k-competitive!

- Proof by trolling.[7] Let's use $k + 1$ pages, and let A be our paging algorithm.

any deterministic algorithm

[7] Common lower bound technique for online algorithms, also commonly used online as well :)

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- **Input sequence:** at each step, request page that A *doesn't have*.

the trolling

we use algorithm is deterministic to construct
the input sequence

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use assumption that we can see the future

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- **Offline Algorithm:** on cache miss, delete page which is requested *furthest in the future*.
- When offline algorithm deletes a page, it's next delete happens after at least k steps.

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Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*

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- Lecture based largely on:
  - Lecture 17 of Luca's Optimization class
  - Lectures 19 and 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Luca's Lecture 17 notes at

  `https://lucatrevisan.github.io/teaching/cs261-11/lecture17.pdf`
- See Karger's Lecture 19 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s22-online.pdf`
- See Karger's Lecture 20 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf`

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