

# Lecture 19: Online Algorithms & Paging

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July 15, 2021

# Overview

- Part I
  - Why Study Online Algorithms?
  - Competitive Analysis
  - Examples
- Paging & Caching
- Conclusion
- Acknowledgements

# Why Study Online Algorithms?

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.

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- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...

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- Applications in
  - Stock Market
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  - many more...
- **Competitive Analysis:** measures performance of our algorithm against best algorithm that could *see into the future* (that is, see the entire input beforehand)<sup>1</sup>
  - ① Worst-case analysis

"offline algorithm"

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- Data Streaming: in this case, we not only receive the input in an online fashion, but we have also *memory constraints*
  - 1 Goal here was to get reasonable (approximate) answers while obeying memory constraints
  - 2 worst-case analysis
- Today, we will only see algorithms which must deal with the input as it receives it, *no constraints in memory*.
  - 1 Goal here is to *be competitive* against *any offline algorithm* (that is, algorithms that could see the entire input beforehand)
  - 2 worst-case analysis



## Competitive Analysis

- Input is given as a sequence  $s = s_1, s_2, \dots, s_n$  of events.

Assume that there is some cost that we are trying to minimize.

Given algorithm  $A$

$$C_A(s) := \text{cost of algorithm } A \text{ on input } s$$

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### Definition (Deterministic Competitive Ratio)

A deterministic online algorithm  $A$  has *competitive ratio*  $k$  (aka  $k$ -competitive) if for all inputs  $s$ , we have:

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### Definition (Randomized Competitive Ratio)

A randomized online algorithm  $A$  has *competitive ratio*  $k$  (aka  $k$ -competitive) if for all inputs  $s$ , we have:

$$\mathbb{E}[C_A(s)] \leq k \cdot C_{opt}(s).$$

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- Having never done this before, we have to decide whether to buy all the equipment or to rent it at the resort.

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If we knew exactly how many times we would go skiing in our lifetime then could not optimally

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- How is this an online algorithm?

*sequence : days of our lives*

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- How is this an online algorithm?
- Each time we go skiing, we have to decide whether to buy or rent (unless we bought it beforehand)
- Algorithm has to decide *when to buy*, knowing only that we have gone skiing  $t$  times

*t-1 times*

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$$t = (\# \text{ times we went skiing in the past}) + 1$$

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- Analysis:
  - If  $t \leq 9$ , then best strategy is to rent: so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot t}{100 \cdot t} = 1$$

OPT: just rent       $C_{opt} = 100 \cdot t$



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- If  $t \geq 10$ , we buy at the 10<sup>th</sup> time, so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot 9 + 1000}{1000} = 1.9$$

$t \geq 10$  OPT: buy in beginning

$C_A =$  cost of renting 9 times + cost of buy

## Secretary Dating Problem

- In the high-tech life, you decide to join a dating site...

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<sup>3</sup>Assumptions: people are comparable AND we know how to do it

<sup>4</sup>Go big or go home lonely!

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- There are  $n$  people that you are interested in dating, and you would like to date the best person<sup>3</sup> out there.<sup>4</sup>

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- **Goal:** maximize probability of dating the best person

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- Consider the following algorithm:

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# Secretary Dating Problem

- Consider the following algorithm:
  - ① Let's assume that all people you want to date are ranked and associate them with their rank:  $1, 2, \dots, n$ 
    - we don't know this rank in advance
    - but we DO know how to compare ranks of two people we have seen  
(relative rank)

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  - 4 After first  $n/e$  dates, you will decide to settle if the person you found is *better* than *anyone else you have dated before*

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- More general algorithm: given a time  $t$ , go on  $t$  dates and from date  $t + 1$  onwards you decide to settle with a person who is better than the previous ones.

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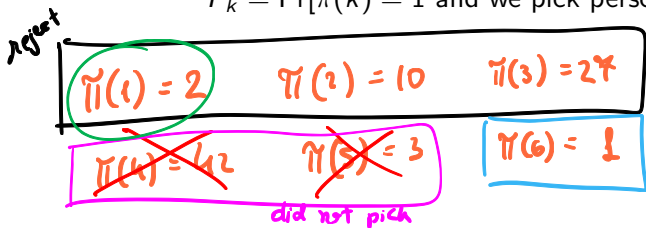
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- Suppose we pick a person at time  $k$ , then want to compute probability

$$P_k = \Pr[\pi(k) = 1 \text{ and we pick person at time } k]$$



$\pi$	1	2	3	4	5	6	...
rank	2	10	27	42	3	1	

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- Then our final success probability will be  $P = \sum_{k>t}^n P_k$
- If  $\pi(k) = 1$ , then  $1 - P_k$  is the probability that we picked a person between  $[t + 1, k - 1]$ , which means someone in this range better than the first  $t$  people. (and the people before it)

$$P_k = \Pr[\underbrace{\pi(k) = 1}_{\text{we did not pick anyone in } [t+1, k-1]} \text{ and } \underbrace{\min \pi(1), \dots, \pi(k-1) \text{ is in } \{\pi(1), \dots, \pi(t)\}}_{\text{min rank before } k \text{ appears in the first } t}]$$

$\Leftrightarrow$  min rank before  $k$  appears in the first  $t$

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$$P_k = \Pr[\pi(k) = 1 \text{ and } \min \pi(1), \dots, \pi(k-1) \text{ is in } \{\pi(1), \dots, \pi(t)\}]$$

$$= \frac{1}{n} \cdot \frac{t}{k-1}$$

$$\Pr[\pi(k) = 1]$$

min of first  $k-1$  dates is in first  $t$  dates

$\Pr[\pi_1 \text{ falls in first } t \text{ positions (out of } k-1) \text{ of random permutation}]$

$$= \frac{t}{k-1}$$

conditioned on  $\pi(k) = 1$

$\pi_1, \pi_2, \dots, \pi_{k-1} \in \{2, 3, \dots, n-1\}$

values that we picked in first  $k-1$  dates

$$\pi_1 < \pi_2 < \pi_3 < \dots < \pi_{k-1}$$

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- From previous slide

$$P_k = \Pr[\pi(k) = 1 \text{ and } \min \pi(1), \dots, \pi(k-1) \text{ is in } \{\pi(1), \dots, \pi(t)\}]$$
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- We get

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$$\sum_{a=1}^n \frac{1}{a} - \sum_{a=1}^t \frac{1}{a}$$



## Secretary Dating Problem

- Final success probability will be  $P = \sum_{k>t}^n P_k$
- From previous slide

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- Wait a second, where is the competitive analysis?

## Making Dating Competitive

- To make the dating problem competitive, we would have to modify it a little bit.
  - We can simply minimize the rank.

*New goal*

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We know  $C_{opt} = 1$

$\therefore$  our algorithm will be  $\Theta(n)$ -competitive

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- That is, as we get older, we become more desperate to find someone and lower our expectations...

- Part I
  - Why Study Online Algorithms?
  - Competitive Analysis
  - Examples
  
- Paging & Caching
  
- Conclusion
  
- Acknowledgements

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How to delete from cache?

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- Simplification: assume we only have cache and main memory.

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- 2 **Random**:  $k$ -competitive
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# LRU Analysis

## Theorem

*For cache of size  $k$ , LRU is  $k$ -competitive.*

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- 1 Upper bound: divide input sequence into phases.
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  - and so on...

algorithm faults  $\leftrightarrow$  cache miss

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*any algorithm must*

- 2 We will prove that OPT algorithm faults *at least* once per phase

*per phase* LRU faults  $k$  times (by definition of phase)

OPT faults  $\geq 1$  time (we'll prove)

$$\Rightarrow C_{LRU} \leq k \cdot C_{OPT}$$

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- 4 Examples of phases, for  $k = 3$ :

1, 1, 2, 2, 1, 3, 4, 3, 2, 4, 5, 6, 15, 4, 4, 2, 3, 5, 6, 4,5

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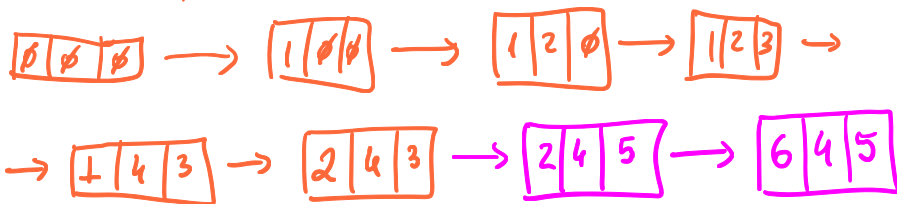
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1 (~~1~~, ~~2~~, ~~2~~, ~~1~~, 3, 4) (3, 2, 4, 5, 6) (15, 4, 4, 2) (3, 5, 6) (4, 5)  
3 faults

# LRU Analysis - Example

Examples of phases, for  $k = 3$ :

$\boxed{1} (\times) \boxed{2}, (\times) \boxed{3}, (\times) \boxed{4} (\times) \boxed{2}, (\times) \boxed{5}, \boxed{6} (15, 4, 4, 2) (3, 5, 6) (4, 5$   
phase 1

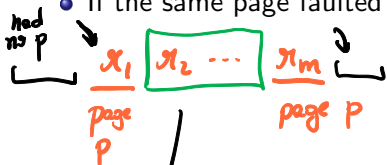


and so on ..

# LRU Analysis - Upper Bound

- Need to prove that OPT will fault at least once per phase.

- If the same page faulted twice in one phase:



within a phase

$\Rightarrow$  must have made  $k+1$  distinct requests in this phase

$\Rightarrow$  Pigeonhole principle tells us that OPT must have faulted ( $\text{OPT has memory} \leq k$ )

must have queried  $k-1$  (different) other pages

in green period P was removed from cache (at  $x_m$  request P and get page fault)

using LRU heuristic P at some point was furthest in past  $\Leftrightarrow$  requested all other  $k-1$  pages after last request for P

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- Proof: Look at last fault page in previous phase.

..... b)  
↖ last fault of previous phase then  
we will have  $b$  in our cache in current phase

- 1) if  $OPT$  faulted at  $b$  then  $OPT$  will have  $b$  in its cache
- 2) if  $OPT$  did not fault, then  $b$  must have been in  $OPT$ 's cache (only way not to fault)

## LRU Analysis - Upper Bound

- If each page faulted once in a phase.
- **Claim:** in the beginning of each phase, content of  $OPT$  and content of our algorithm  $A$  intersect in at least one page.
- Since  $OPT$  and  $A$  had a common page, then  $OPT$  must have faulted as well (since each page faulted in this phase)

in particular  $b$  also faulted.

concludes our proof of  $\leq k$ -competitive

## Lower Bound - Deterministic Paging Algorithms

### Theorem

*Any deterministic algorithm for paging with  $k$  pages is at least  $k$ -competitive!*

- Proof by trolling.<sup>7</sup> Let's use  $k + 1$  pages, and let  $A$  be our paging algorithm.

*any deterministic algorithm*

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*the trolling*

*we use algorithm is deterministic to construct  
the input sequence*

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- When offline algorithm deletes a page, it's next delete happens after at least  $k$  steps.

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# Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*

# Acknowledgement

- Lecture based largely on:
  - Lecture 17 of Luca's Optimization class
  - Lectures 19 and 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Luca's Lecture 17 notes at  
<https://lucatrevisan.github.io/teaching/cs261-11/lecture17.pdf>
- See Karger's Lecture 19 notes at  
<http://courses.csail.mit.edu/6.854/06/scribe/s22-online.pdf>
- See Karger's Lecture 20 notes at  
<http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf>

# References I

-  [Motwani, Rajeev and Raghavan, Prabhakar \(2007\)](#)  
Randomized Algorithms