Lecture 18: Hardness of Approximation

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Overview

- Background and Motivation
  - Why Hardness of Approximation?
  - How do we prove Hardness of Approximation?
  - Hardness of Approximation - Example
- Proofs & Hardness of Approximation
- Conclusion
- Acknowledgements

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others

John Nash

Gödel → Von Neumann (letter)

Johnson, Garey and others

in TCS and CO

combinatorial optimization problems seem to be intractable

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"heuristics"

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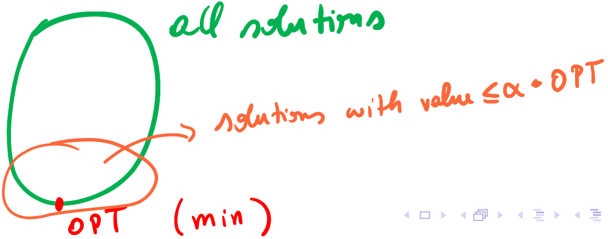

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Hardness of Approximation

Reduction: α-approximation of $\mathcal{P}$ $\Leftrightarrow$ exactly solving $\mathcal{P}$

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- Important to know the limits of efficient algorithms!

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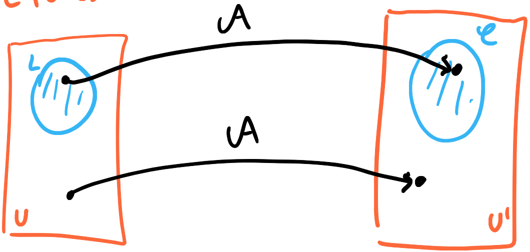
in pictures : blue region are YES inputs



(size of the reduction is small)

if we had an algorithm that solves C then we could also solve L.

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  - maps every YES instance of SAT to a YES instance of CLIQUE
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If φ is a boolean formula, then map φ to graph (G_φ, k) that has a k-clique if φ is satisfiable

If φ is NOT satisfiable, then we have to map $\varphi \mapsto (G_\varphi, k)$ where G_φ has no clique of size k

G_φ has clique of size $\leq k-1$ (NO instance)

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$\varphi \overset{A}{\longmapsto} \begin{cases} (G_\varphi, k) & \text{YES of Clique } (\varphi \text{ is satisfiable}) \\ (H_\varphi, k) & \text{VERY MUCH NO} \quad H_\varphi \text{ does not have cliques of size } k/3 \end{cases}$

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Suppose we had a 2-approximation for MAX-CLIQUE
(if input G has clique of size k we output clique of size $\geq k/2$)

MAX-Clique$(G_\varphi) \longmapsto$ return clique of size $\geq k/2$
MAX-Clique$(H_\varphi) \longmapsto$ return clique of size $\leq k/3$

our alg. would be able to distinguish if φ is satisfiable

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- One of the famous NP-complete problems

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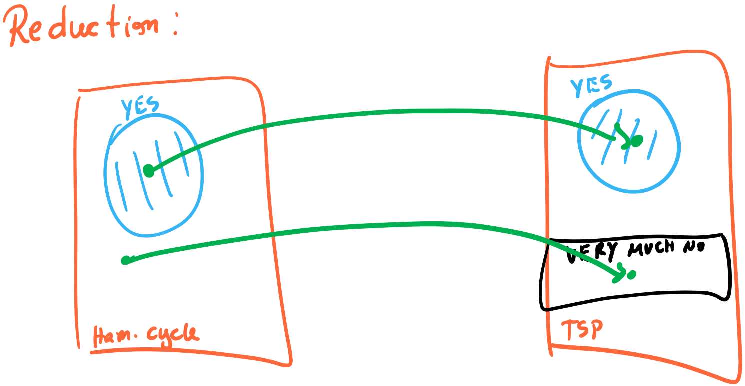
3. In our case, let's reduce it to the *Hamiltonian Cycle Problem*

Theorem

If there is an algorithm M which solves TSP without repetitions with α-approximation, then $P = NP$.

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex ~~at most~~ exactly once.



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H complete graph on V

(weighted)

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    - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \\ (1 + \alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \end{cases}$

small weight

very large weight

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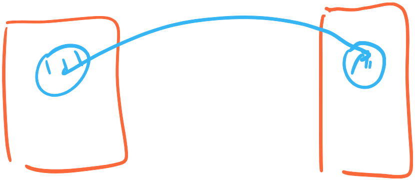
3. If G has a Hamiltonian Cycle, then OPT for the TSP is of value $\leq |V|$

only use edges of weight 1



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gap between YES and NO instances

if G has no Hamiltonian cycle then TSP has only OPT of length $\geq (1+\alpha) \cdot |V|$

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4. If G has no Hamiltonian Cycle, then OPT for TSP must use an edge not in V, thus value is $\geq (1+\alpha) \cdot |V|$
5. Thus, M on input H will output a Hamiltonian Cycle of G, if G has one, or it will output a solution with value $\geq (1+\alpha) \cdot |V|$

Discussion of Proof

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Complexity Classes

- **NP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine V, such that:

$$x \in L \Leftrightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)} \text{ s.t. } V(x, w) = 1$$

verifier

witness (or proof) of poly size

V accepts iff w is a valid "proof" that x ∈ L

Hamiltonian cycle

x : G(V, E)

w ← is a Hamiltonian cycle

V(x, w) checks whether w is Hamiltonian cycle

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$$x \in L \Leftrightarrow \Pr_{R \in \{0,1\}^{\text{poly}(|x|)}} [M(x,R) = 1] \geq 2/3$$

over choice of random strings

accepts with high probability iff $x \in L$

Remark: a randomized algorithm in terms of TM is Turing machine $M(\cdot,\cdot)$

x regular input

random coins

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if $x \in L$ accept with high probability

$$x \in L \Rightarrow \Pr_{R \in \{0,1\}^{\mathsf{poly}(|x|)}}[M(x,R) = 1] \geq 2/3$$

if $x \notin L$ never accept

$$x \notin L \Rightarrow \Pr_{R \in \{0,1\}^{\mathsf{poly}(|x|)}}[M(x,R) = 1] = 0$$

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- **co-RP:** languages $L \subseteq \{0,1\}^*$ s.t. $\overline{L} \in RP$

Examples of Problems in Complexity Classes

Polynomial Identity Testing problem:

given polynomial $p(\bar{x})$ of degree 1^d ($\underbrace{11\cdots1}_{d \text{ times}}$)

is $p \equiv 0$?

YES: only zero polynomial

NO: nonzero polynomial

Claim: PIT $\in$ co-RP

algorithm M(polynomial, evaluation point)

$M(p, \alpha) = 1 \iff p(\alpha) = 0$

Schwarz-Zippel alg.

$p \in L \Rightarrow \Pr_R[M(p, R) = 1] = 1$

$p \notin L \Rightarrow \Pr_R[M(p, R) = 0] \geq 2/3$

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    - Verifier picks a poly-time Turing Machine V and outputs $\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$

NP as Proof System - Example

input: $x \leftarrow G(V, E)$

prover: wants to claim that G has Ham. cycle

↳ give proof w

verifier: $M(x, w)$

checks whether w is ham. cycle for x

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   of verifier

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    $$\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$$
    there is valid proof
    - **Completeness:** $x \in L \Rightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)}$ such that $V(x,w) = 1$

"x correct statement"

proof valid if verifier accepts it

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0,1\}^{\text{poly}(|x|)}$
   - Verifier picks a deterministic, poly-time Turing Machine V and outputs $\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$
   - **Completeness:** $x \in L \Rightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)}$ such that $V(x,w) = 1$
   - **Soundness:** $x \notin L \Rightarrow \forall w \in \{0,1\}^{\text{poly}(|x|)}$ we have $V(x,w) = 0$

false statement — any proof — not valid

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Probabilistic Proof Systems

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In a probabilistic proof system, the verifier has a randomized algorithm V for which:

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1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V^w(x) = 1] = 1$

verifier V on oracle access to w

V can query bits of w

∃ proof w

always accepts correct statement

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Can model randomized algorithm V as

Turing Machine $M^w(x, R)$

$x \in L \Rightarrow \exists w$ s.t. $\Pr_R[M^w(x,R) = 1] = 1$

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OBS: V is poly-time in x (x is input, w only oracle access)

Interactive Proofs: Complexity Classes

The above discussion motivates us to define complexity classes in terms of proof systems!

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The class IP consists of all languages L that have an interactive proof system (P, V) where

prover

verifier

prover : algorithm that outputs proofs

Interactive Proofs: Complexity Classes

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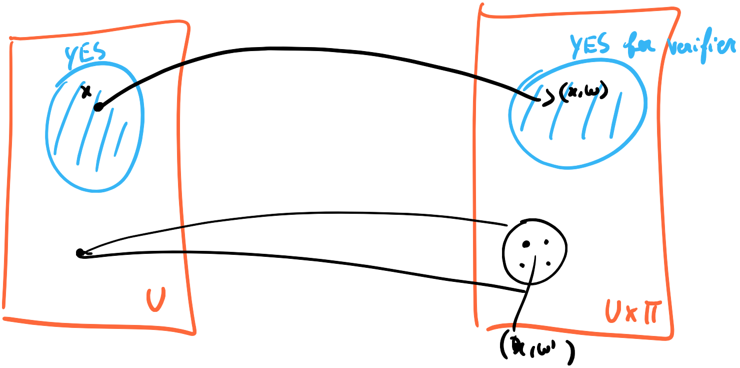
1. the verifier V is a randomized, polynomial time algorithm
2. there is an honest prover P (who can be all powerful)
3. for any $x \in \{0,1\}^*$
    - $x \in L \Rightarrow$ for an *honest* prover P, the proof Π_P satisfies:

$$\Pr[V^{\Pi_P}(x) = 1] = 1$$

    - $x \notin L \Rightarrow$ for *any prover* P', the proof $\Pi_{P'}$

$$\Pr[V^{\Pi_{P'}}(x) = 1] \leq 1/2$$

Interactive Proofs - Picture



Quantifying Probabilistic Proof Systems

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- $PCP[r(n), q(n)]$ consists of all languages $L \in PCP$ such that, on inputs x of length n

set of languages

L has prob. check. proof

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- $PCP[r(n), q(n)]$ consists of all languages $L \in PCP$ such that, on inputs x of length n
  1. Uses $O(r(n))$ random bits
  2. Examines $O(q(n))$ bits of a proof w

n only depends on x

verifier

Note that n *does not* depend on w, only on x.

$M^w(x, R)$

$|R| = O(r(n))$

bits

M only makes $O(q(n))$ queries to proof w.

M running time poly(n)

Quantifying Probabilistic Proof Systems

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$O(\log n)$ randomness

$O(1)$ bits of proof.

Note that n *does not* depend on w, only on x.

Theorem (PCP theorem [AS'98, ALMSS'98])

$$PCP[\log n, 1] = NP$$

PCP and Approximability of Max 3SAT

Definition (Max 3SAT)

- **Input:** a 3CNF formula φ on boolean variables $x_1, \ldots, x_n$ and m clauses
- **Output:** the maximum number of clauses of φ which can be simultaneously satisfied.

Theorem

1. *The PCP theorem implies that there is an $\varepsilon > 0$ such that there is no polynomial time $(1 + \varepsilon)$-approximation algorithm for Max 3SAT, unless $P = NP$.*
2. *Moreover, if Max 3SAT is hard to approximate within a factor of $(1 + \varepsilon)$, then the PCP theorem holds.*

- In other words, the PCP theorem and the hardness of approximation of Max 3SAT are equivalent.

PCP and Approximability of Max 3SAT

1. Let us assume the PCP theorem holds. $PCP[\log n, 1] = NP$
   - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
   - Let V be the $(O(\log n), q)$ verifier for L, where q is a constant

PCP and Approximability of Max 3SAT

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2. We now describe a reduction from L to Max 3SAT which has a gap.

if we do (2) we have hardness of approximation

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3. Given an instance x of problem L, we construct 3CNF formula φ_x with m clauses such that, for some ε we have
   - $x \in L \Rightarrow \varphi_x$ is satisfiable
   - $x \notin L \Rightarrow$ no assignment satisfies more than $(1 - \varepsilon) \cdot m$ clauses of φ_x

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4. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n). $2^{O(\log n)}$
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.

R is our random string

bits of proof w

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2. Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
   - Can be done with a CNF of size 2^q
   - Converting to 3CNF we get a formula of size $q \cdot 2^q$

any boolean function on $\{0,1\}^q \to \{0,1\}$ can be expressed as a CNF.

usual reduction from CNF to 3CNF

PCP and Approximability of Max 3SAT

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3. Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above

PCP and Approximability of Max 3SAT

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   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
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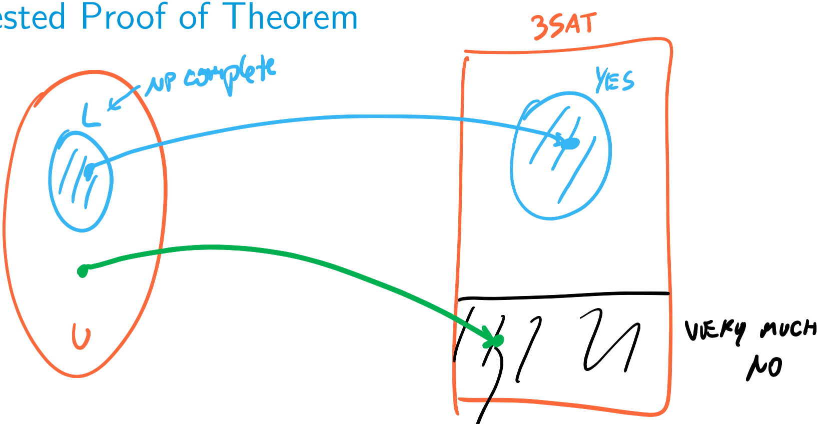
   for each R
3. Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above
4. If $x \in L$ then there is a witness w such that $V(x, w)$ accepts for every random string R. In this case, φ_x is satisfiable!
5. If $x \notin L$ then the verifier says NO for half of the random strings R.
   - For each such random string, at least one of its clauses fails
   - Thus at least $\varepsilon = \dfrac{1}{2 \cdot q \cdot 2^q}$ of the clauses of φ_x fails.

   half of random strings

Digested Proof of Theorem



USED PCP theorem to construct the efficient reduction with "large" gap

Q $\frac{1}{2q2^q}$ - fraction of clauses will ALWAYS be violated

Digested Proof of Theorem

YES $\rightarrow$ all clauses satisfiable

1 - fraction

NO $\rightarrow$ $\leq \left(1 - \frac{1}{2q2^q}\right)$ fraction of clauses satisfiable

$\underbrace{\hphantom{2q2^q}}_{\epsilon}$

$\therefore$ MAX 3SAT $(1+\epsilon)$ hard to approximate

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems
- Proof systems, in particular *Probabilistic Checkable Proofs*, allows us to get such strong reductions
- Many more applications in computer science and industry!
  - Program Checking (for software engineering)
  - Zero-knowledge proofs in cryptocurrencies
  - many more...

Acknowledgement

- Lecture based largely on:
  - Section's 1-3 of Luca's survey [Trevisan 2004]
  - [Motwani & Raghavan 2007, Chapter 7]
- See Luca's survey `https://arxiv.org/pdf/cs/0409043`

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