Lecture 17: Semidefinite Programming Relaxation and MAX-CUT

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Overview

- Why Relax & Round?
- Max-Cut SDP Relaxation and Rounding
- Conclusion
- Acknowledgements

Motivation - NP-hard problems

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- What do we do when we see one?
  1. Find approximate solutions in polynomial time!
  2. Sometimes we even do that for problems in P (but we want much much faster solutions)

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  1. Find approximate solutions in polynomial time!

- **Integer Linear Program (ILP):**

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \in \mathbb{N}^n \end{aligned}$$

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- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

Motivation - NP-hard problems

- **Quadratic Program (QP):**

$$\begin{aligned} \text{minimize} \quad & g(x) \\ \text{subject to} \quad & q_i(x) \geq 0 \quad \} \ 1 \leq i \leq m \end{aligned}$$

where each $q_i(x)$ and $g(x)$ are quadratic functions on x.

Constraint   objective function

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linear constraint also quadratic (LP ⊂ QP)

$x_i \in \{0,1\} \Leftrightarrow x_i(1-x_i) = 0$ quadratic

enough to capture NP-hard problems

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Analogue to : relax ILPs with LPs

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  - Can we get **better** approximations using SDPs instead of ILPs?
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- Very impressive recent theoretical developments! Unique Games Conjecture, Sum-of-Squares, and more!

Example

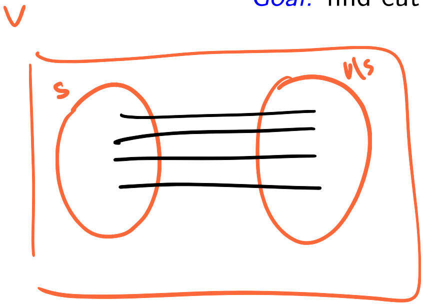
Maximum Cut (Max-Cut):

$$G(V, E) \text{ graph.}$$

$$\text{Cut } S \subseteq V \text{ and size of cut is}$$

$$|E(S, \overline{S})| = |\{(u, v) \in E \mid u \in S, v \notin S\}|.$$

Goal: find cut of maximum size. NP-hard!



Example

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Cut $S \subseteq V$ and size of cut is

$$|E(S,\overline{S})| = |\{(u,v) \in E \;\mid\; u \in S, v \notin S\}|.$$

Goal: find cut of maximum size.

$x_u = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{otherwise} \end{cases}$ $\quad z_e = \begin{cases} 1 & \text{if edge } e \in E(S,\bar{S}) \\ 0 & \text{otherwise} \end{cases}$

Integer Linear Program:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} z_e \\ \text{subject to} \quad & x_u + x_v \geq z_e \quad \text{for } e = \{u,v\} \in E \\ & 2 - x_u - x_v \geq z_e \quad \text{for } e = \{u,v\} \in E \\ & x_v \in \{0,1\} \quad \text{for } v \in V \end{aligned}$$

maximize size of cut

$z_e = 1 \Rightarrow$ one of x_u, x_v must be 1

$z_e = 1 \Rightarrow$ one of x_u, x_v must be 0

choose cut

if $x_u = x_v$ (u, v in the same side of cut) $\Leftrightarrow e \notin E(S,\bar{S})$ $\hookrightarrow \{u,v\}$

$e \in E(S,\bar{S}) \Leftrightarrow x_u + x_v = 1$

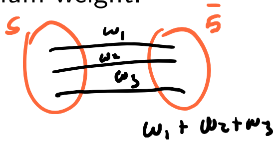
Example - Weighted Variant

Maximum Cut (Max-Cut):

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Cut $S \subseteq V$ and weight of cut is the sum of weights of edges crossing cut.
Goal: find cut of maximum weight.

Integer Linear Program:



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unchanged

Relax... & Round!

In our quest to get efficient (exact or approximate) algorithms for problems of interest, the following strategy is very useful:

[1] Even more general mathematical program, so long as derive SDP from it.

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2. Derive SDP from the QP by going to higher dimensions and imposing PSD constraint

This is called an *SDP relaxation*.

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3. We are still maximizing the same objective function, but over a (potentially) larger set of solutions.

$$OPT(SDP) \geq OPT(QP)$$

(if we are maximizing)

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   1. If solution to SDP is *integral* and *one-dimensional*, then it is a solution to QP and we are done
   2. If solution has *higher dimension*, then we have to devise *rounding procedure* that transforms

      high dimensional solutions $\rightarrow$ integral & 1D solutions

      rounded SDP solution value $\geq c \cdot OPT(QP)$

[1] Even more general mathematical program, so long as derive SDP from it.

Analyzing ILP for Max-Cut

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

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Analyzing ILP for Max-Cut

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Integer Linear Program: $w_e > 0$

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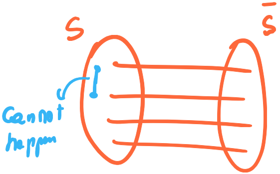
- $OPT(ILP) = 1 \Leftrightarrow G$ is bipartite



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- $OPT(ILP) = 1 \Leftrightarrow G$ is bipartite
- $OPT(ILP) \geq 1/2$

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\end{aligned}
$$

- $OPT(ILP) = 1 \Leftrightarrow G$ is bipartite
- $OPT(ILP) \geq 1/2$
- G complete graph $\Rightarrow OPT = \frac{1}{2} + \frac{1}{2(n-1)}$

weights are uniform

Analyzing ILP for Max-Cut

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- $OPT(ILP) \geq 1/2$
- G complete graph $\Rightarrow OPT = \frac{1}{2} + \frac{1}{2(n-1)}$
- Max-Cut NP-hard

Proof that $OPT(ILP) \geq 1/2$

Probabilistic method:

Pick : $x_v = \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$

$z_{uv} = 1 \iff \begin{matrix} x_u = 1 & x_v = 0 & 1/4 \\ x_u = 0 & x_v = 1 & 1/4 \end{matrix}$

$\mathbb{E}[z_{uv}] = \frac{1}{2}$

$$\mathbb{E}[\text{value of cut}] = \sum_{e \in E} w_e \cdot \underbrace{\mathbb{E}[z_e]}_{1/2} = \frac{1}{2} \sum_{e \in E} w_e = \frac{1}{2}$$

$\therefore \exists$ integral solution (cut) that has value

$\geq$ average (expectation) $\quad \therefore \; OPT(ILP) \geq 1/2$

for any graph

Rounding Max-Cut ILP

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Linear Program Relaxation:

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} z_e \cdot w_e \\ \text{subject to} \quad & x_u + x_v \geq z_e \quad \text{for } e = \{u, v\} \in E \\ & 2 - x_u - x_v \geq z_e \quad \text{for } e = \{u, v\} \in E \\ & 0 \leq x_v \leq 1 \quad \text{for } v \in V \\ & 0 \leq z_e \leq 1 \quad \text{for } e \in E \end{aligned}$$

LP relaxation

Rounding Max-Cut ILP

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- Setting $x_v = 1/2$, $z_e = 1$ we get $OPT(LP)$ always $= 1$

Rounding Max-Cut ILP

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- Setting $x_v = 1/2$, $z_e = 1$ we get $OPT(LP)$ always $= 1$
- This relaxation is not helpful! :(

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Max-Cut

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

$e \in E(S, \bar{S})$ iff $x_u \neq x_v$

iff $x_u x_v = -1$

Quadratic Program:

$$\text{maximize} \quad \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - x_u x_v)$$

$$\text{subject to} \quad x_v^2 = 1 \quad \text{for } v \in V$$

$\frac{1}{2}(1 - x_u x_v) = \begin{cases} 1 & \text{if } e \in E(S, \bar{S}) \\ 0 & \text{o.w.} \end{cases}$

$x_u \in \{+1, -1\}$

$+1 \to \in S$, $-1 \to \notin S$

$S = \{u \mid x_u = 1\}$

$\bar{S} = \{v \mid x_v = -1\}$

ILP: maximize $\sum w_e z_e$

s.t. $z_e \leq x_u + x_v$ $\quad e = \{u, v\}$

$z_e \leq 2 - x_u - x_v$

$x_u \in \{0, 1\}$

SDP Relaxation [Delorme, Poljak 1993]

$G(V, E, w)$ weighted graph, $|V| = n$ and $\sum_{e \in E} w_e = 1$

Semidefinite Program:

captures $1 - x_u x_v$

$$\text{maximize} \quad \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - y_u^T y_v\right)$$

$$\text{subject to} \quad \|y_v\|_2^2 = 1 \quad \text{for } v \in V$$

← captures $x_v^2 = 1$

$$y_v \in \mathbb{R}^d \quad \text{for } v \in V$$

higher dimension

QP: $\max \sum \frac{1}{2} w_{u,v} (1 - x_u x_v)$

s.t. $x_v^2 = 1$

$\|y_v\|_2 = \sqrt{y_v(1)^2 + y_v(2)^2 + \dots + y_v(d)^2}$

OBS: if in the above program $y_v(j) = 0$ for all $j > 1$ then we recover the QP

SDP Relaxation [Delorme, Poljak 1993]

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$$\text{subject to} \quad \|y_v\|_2^2 = 1 \quad \text{for } v \in V$$

$$y_v \in \mathbb{R}^d \quad \text{for } v \in V$$

linear in entries of X; X_{uv}; enough to take $d = n$

- How is that an SDP?

Let X $n \times n$ matrix (symmetric) where $X_{uv} = y_u^T y_v$

($X_{vu} = y_v^T y_u = y_u^T y_v$)

$\therefore X = Y^T Y$ where $Y = (y_1 \; y_2 \cdots y_n)$

$\therefore$ $X \succeq 0$ — constraints

$y_v^T y_v = \|y_v\|_2^2 = 1$ ($= X_{vv}$)

$X_{vv} = 1 \quad \forall v \in V$ — norm constraint

SDP formulation

$$\text{maximize} \sum_{\{u,v\} \in E} \frac{1}{2} w_{u,v} \cdot (1 - X_{uv})$$

$$\text{s.t.} \quad X_{vv} = 1 \quad \forall v \in V$$

$$X \succeq 0$$

(in primal form!)

What is this SDP doing?

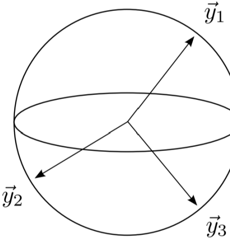

Figure 10.1: Vectors $\vec{y}_v$ embedded onto a unit sphere in $\mathbb{R}^d$.

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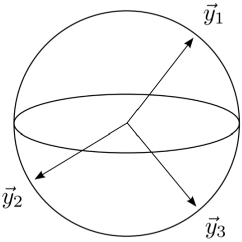
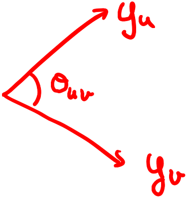


Figure 10.1: Vectors $\vec{y}_v$ embedded onto a unit sphere in $\mathbb{R}^d$.

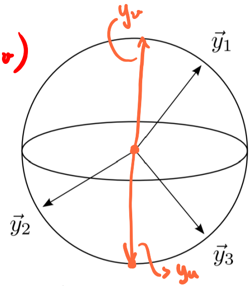
- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$

$$y_u^T y_v = \underbrace{\|y_u\| \cdot \|y_v\|}_{=1} \cdot \cos(\theta_{u,v})$$



What is this SDP doing?

$\max \sum \frac{1}{2} w_{u,v} (1 - \gamma_{uv})$, $\gamma_{uv} = y_u^T y_v$



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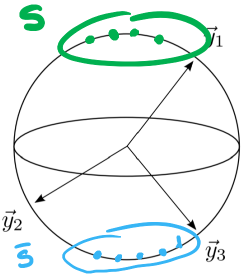
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight

$\gamma_{uv} = -1$ iff y_u, y_v are antipodes in sphere

What is this SDP doing?



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- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight
- Geometrically, want vertices from our max-cut S to be as far away from the complement $\overline{S}$ as possible

What is this SDP doing?

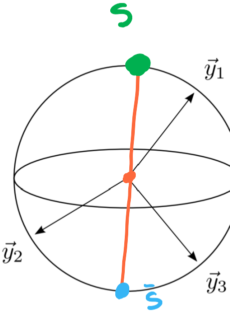


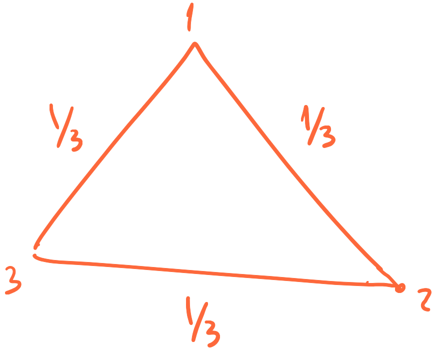
Figure 10.1: Vectors $\vec{y}_v$ embedded onto a unit sphere in $\mathbb{R}^d$.

- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight
- Geometrically, want vertices from our max-cut S to be as far away from the complement $\overline{S}$ as possible
- If all y_v's are in a one-dimensional space, then we get original quadratic program

$$OPT(SDP) \geq \text{Weight of Maximum Cut}$$

Example

Let's consider $G = K_3$ with equal weights on edges.



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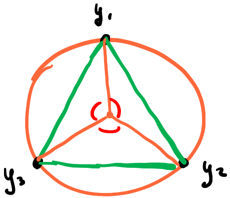
- Embed $y_1, y_2, y_3 \in \mathbb{R}^2$ 120 degrees apart in unit circle
- We get:

all angles are 120°



weight

$$OPT(SDP) = \sum_{i<j} \frac{1}{2} \cdot \frac{1}{3} \left(1 - \cos\left(\frac{2\pi}{3}\right)\right)$$

$$= 3 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \left(1 + \frac{1}{2}\right)$$

$$= 3/4$$

max cut $= OPT(QP) = \frac{2}{3}$

OPT(SDP) can be strictly larger than OPT of QP

Example

Let's consider $G = K_3$ with equal weights on edges.

- Embed $y_1, y_2, y_3 \in \mathbb{R}^2$ 120 degrees apart in unit circle
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- $OPT_{SDP}(K_3) = 3/4$
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- So we get approximation $8/9$ (better than the LP relaxation)

LP get $2/3$ approximation

$(OPT_{LP} = 1)$

$OPT_{LP} > OPT_{SDP} > OPT_{\text{max-cut}}$

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- **Practice problem:** try this with C_5.

should get ≈ 0.88

Max-Cut - Rounding

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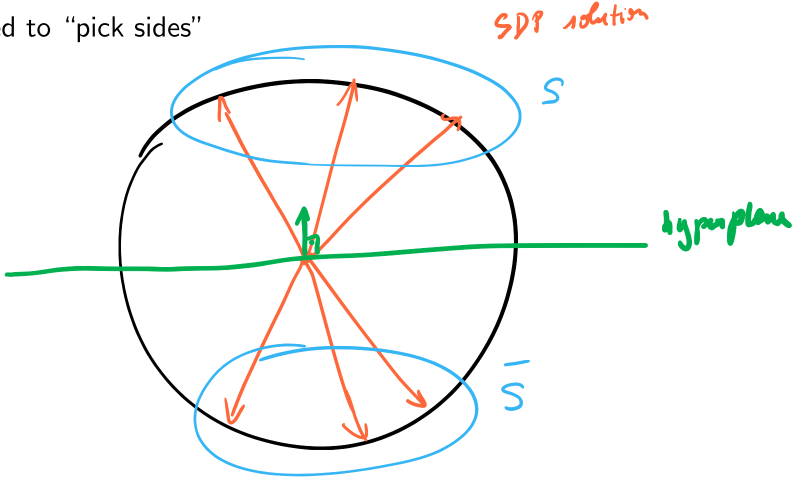

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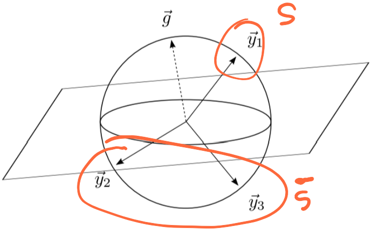

Figure 10.2: Vectors being separated by a hyperplane with normal $\vec{g}$.

Analysis of Rounding - Sketch

- Probability that edge $\{u, v\}$ crosses the cut is same as probability that z_u, z_v fall in different sides of hyperplane

$$\Pr[\{u, v\} \text{ crosses cut }] = \Pr[g \text{ splits } z_u, z_v]$$

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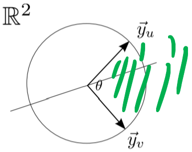

Figure 10.3: The plane of two vectors being cut by the hyperplane

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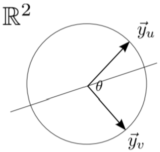

Figure 10.3: The plane of two vectors being cut by the hyperplane

- Probability of splitting z_u, z_v:

$$\Pr[\{u, v\} \text{ crosses cut}] = \frac{\theta}{\pi} = \frac{\cos^{-1}(z_u^T z_v)}{\pi} = \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

$$\mathbb{E}[\text{value of cut}] = \sum_{\{u,v\} \in E} w_{u,v} \cdot \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

Analysis of Rounding - Sketch

- Expected value of cut:

assume that we have concentration

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Analysis of Rounding - Sketch

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$$OPT_{SDP} = \sum_{\{u,v\}\in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - z_u^T z_v\right) = \sum_{\{u,v\}\in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - \gamma_{uv})$$

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Theorem ([Goemans, Williamson 1994])

$\alpha = 0.87856...$ *works, and gives us our approximation algorithm.*

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4. Solve SDP optimally using efficient algorithm.
   1. If solution to SDP is *integral* and *one dimensional*, then it is a solution to Max-Cut and we are done
   2. If have *higher dimensional* solutions, *rounding procedure*

      Random Hyperplane Cut algorithm, with high probability we get

$$OPT_{\text{max-cut}} \geq cost(\text{rounded solution}) \geq 0.878 \cdot OPT(SDP) \geq 0.878 \cdot OPT(\text{Max-Cut})$$

Remarks

1. SDPs are very powerful for solving (approximating) many hard problems

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All of these are amazing final project topics!

Conclusion

- Mathematical programming - very general, and pervasive in (combinatorial) algorithmic life
- Mathematical Programming hard in general
- Sometimes can get SDP rounding!
- Solve SDP and round the solution
  - Deterministic rounding when solutions are nice
  - Randomized rounding when things a bit more complicated

Max-cut

Acknowledgement

- Lecture based largely on:
  - Lecture 14 of Anupam Gupta and Ryan O'Donnell's Optimization class

    https://www.cs.cmu.edu/~anupamg/adv-approx/
- See their notes at

  https://www.cs.cmu.edu/~anupamg/adv-approx/lecture14.pdf

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