Lecture 14: Linear Programming Relaxation and Rounding

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Overview

- Part I
  - Why Relax & Round?
- Vertex Cover
- Set Cover
- Conclusion
- Acknowledgements

Motivation - NP-hard problems

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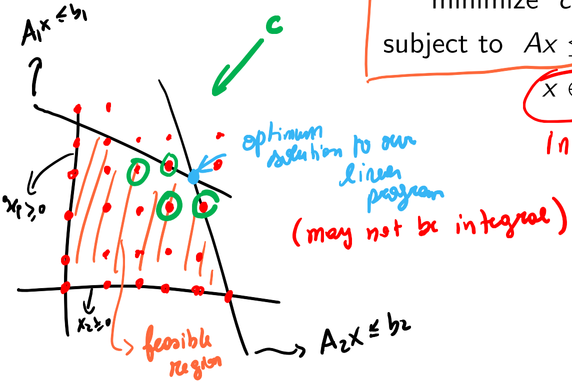
- **Integer Linear Program (ILP):**

$$\text{minimize} \quad c^T x$$
$$\text{subject to} \quad Ax \le b$$
$$x \in \mathbb{N}^n$$

LP

Integrality constraints



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- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

Example

NP-hard

Maximum Independent Set:

input: $G(V, E)$ graph.

Independent set $S \subseteq V$ such that $u, v \in S \Rightarrow \{u, v\} \notin E$.

not connected by edge

Integer Linear Program:

$$\text{maximize} \quad \sum_{v \in V} x_v \quad = \text{size of } S$$

$$\text{subject to} \quad x_u + x_v \leq 1 \quad \text{for } \{u, v\} \in E$$

$$x_v \in \{0, 1\} \quad \text{for } v \in V$$

if $\{u, v\} \in E$ then at most one of u or v belongs to S

$$x_v = \begin{cases} 0 & \text{if } v \notin S \\ 1 & \text{if } v \in S \end{cases}$$

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3. We are still minimizing the same objective function, but over a (potentially) larger set of solutions.

$$opt(LP) \leq opt(ILP)$$

because the LP has less constraints than the ILP

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   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If solution has *fractional values*, then we have to devise *rounding procedure* that transforms

fractional solutions $\rightarrow$ integral solutions

min $\alpha^T x$

$z \longmapsto \hat{z}$ integral

$$opt(LP) \le c \cdot opt(ILP)$$

$\alpha^T z = opt(LP) \longmapsto \alpha^T \hat{z} \le c \cdot OPT(ILP)$

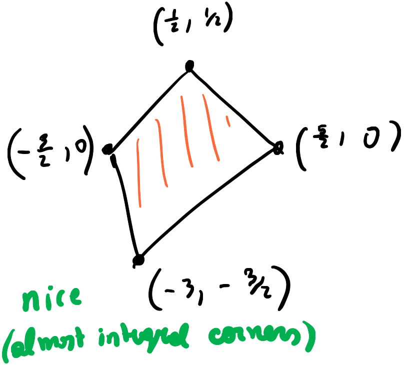
$\alpha^T \hat{z} \le c \cdot \alpha^T z$
$= c \cdot OPT(LP)$
$\le c \cdot OPT(ILP)$

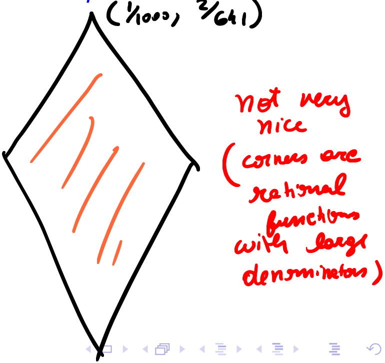
Not all LPs created equal

When solving LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

it is important to understand *geometry of feasible set* & how nice the *corner points* are, as they are the candidates to *optimum* solution.





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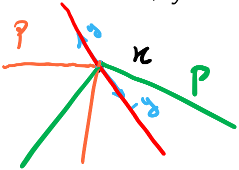
- Let $P := \{x \in \mathbb{R}^n_{\geq 0} \mid Ax = b\}$
- **Vertex Solutions:** a solution $x \in P$ is a vertex solution if $\nexists y \neq 0$ such that $x + y \in P$ *and* $x - y \in P$



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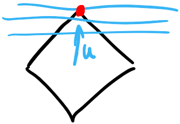

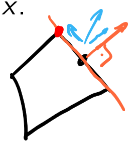

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Practice problem:
all three definitions are equivalent!

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- **Basic Solutions:** let $supp(x) := \{i \in [n] \mid x_i > 0\}$ be the set of nonzero coordinates of x. Then $x \in P$ is a basic solution $\Leftrightarrow$ the columns of A indexed by $supp(x)$ are linearly independent.

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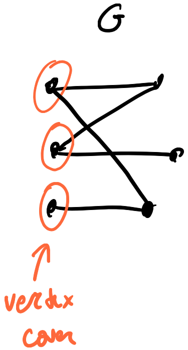
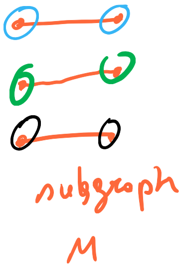
- **Output:** Minimum number of vertices that "touches" all edges of graph. That is, minimum set S such that for each edge $\{u, v\} \in E$ we have

$$|S \cap \{u, v\}| \geq 1.$$





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total cost

form a cover of all edges

$\hookrightarrow x_u = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S \end{cases}$

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add both u, v to my set S

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- By construction, S is a vertex cover.
- If added elements to S k times, then $|S| = 2k$ and G has a matching of size k, which means that optimum vertex cover is at least k.

$\{u_1, v_1\}$ $\{u_2, v_2\}$ $\{u_3, v_3\}$ . . . $\{u_k, v_k\}$

matching of G of size k

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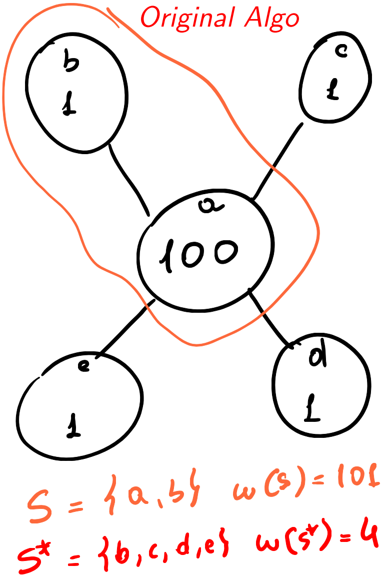
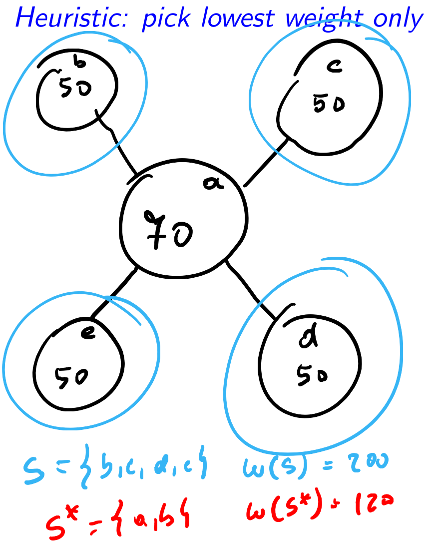
- Thus, we get a 2-approximation.

What can go wrong in the weighted case?





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hard constraint

2. Drop integrality constraints

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new inequalities

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4. Round LP as follows: round z_v to nearest integer.

Vertex Cover - Analysis

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3. Round z_v to nearest integer. That is $y_v = \begin{cases} 1, & \text{if } z_v \geq 1/2 \\ 0, & \text{if } 0 \leq z_v < 1/2 \end{cases}$

↓ rounded solution

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$z_u + z_v \geq 1 \Rightarrow$ one of z_u, z_v is $\geq 1/2$

$\Rightarrow$ one of y_u, y_v must be 1

y solution to ILP!

Vertex Cover - Analysis

$y_v = 1 \Rightarrow z_v \geq 1/2 \Rightarrow 2z_v \geq 1 = y_v$

$y_v = 0 \Rightarrow z_v \geq 0 \Rightarrow 2z_v \geq y_v$

$\therefore \boxed{y_v \leq 2z_v}$

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6. Cost of y is:

$$\sum_{u \in V} c_u \cdot y_u \leq \sum_{u \in V} c_u \cdot (2 \cdot z_u) \leq 2 \cdot OPT(ILP)$$

$= 2 \cdot OPT(LP) \leq$

Set Cover

Setup:

- **Input:** a finite set U and a collection $S_1, S_2, \ldots, S_n$ of subsets of U.
- **Output:** The fewest collection of sets $I \subseteq [n]$ such that

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minimize weight of our collection

collection must cover U

pick or don't pick set S_i

Set Cover - Relax...

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3. Can we just round each coordinate z_i to the nearest integer (like in vertex cover)?

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1. Obtain LP relaxation:

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in [n]} w_i \cdot x_i \\ \text{subject to} \quad & \sum_{i : v \in S_i} x_i \geq 1 \quad \text{for } v \in U \\ & 0 \leq x_i \leq 1 \quad \text{for } i \in [n] \end{aligned}$$

2. Suppose we end up with fractional solution $z \in [0,1]^n$ when we solve the LP above. Now need to come up with a rounding scheme.
3. Can we just round each coordinate z_i to the nearest integer (like in vertex cover)?
4. Not really. Say $v \in U$ is in 20 sets, and we got $z_i = 1/20$ for each of the sets $v \in S_i$. Then rounding procedure above would not select any such set!

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don't pick S_i with prob. $1 - z_i$

independent for each $i \in [n]$

pick S_i $B(z_i)$

$z_i \in [0,1]$ for each i.

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4. Expected cost of the sets is $\sum_{i=1}^{n} w_i \cdot z_i$, which is the optimum for the LP. But will this process cover U?

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- What is probability that v is covered in Random Pick?

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  $v \in S_1, S_2 \quad z_1 = z_2 = 1/2$

  $\Pr[\text{not cover } v] = \underbrace{\Pr[\text{not pick } S_1]}_{1/2} \cdot \underbrace{\Pr[\text{not pick } S_2]}_{1/2}$

  $= 1/4$

- Definitely not 1. Think about case $k = 2$ and $z_1 = z_2 = 1/2$.

  $\Pr[\text{cover } v] = 3/4$.

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$$\mathbb{E}[\# \text{ uncovered elements}] = \frac{1}{k} \cdot |U|$$

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- By perseverance! :)

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Lemma (Probability of Covering an Element)

In a sequence of k independent experiments, in which the i^{th} experiment has success probability p_i, and

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- Thus probability of failure is

$-\sum p_i \leq -1$

$$\prod_{i=1}^{k} (1 - p_i) \leq \prod_{i=1}^{k} e^{-p_i} = e^{-p_1 - \cdots - p_k} \leq 1/e$$

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Random pick procedure

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To analyze this, need to show that we don't execute the for loop too many times.

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$$X = t \cdot \sum_{i=1}^{k} w_i \cdot z_i$$

- By Markov:

$$\Pr[X \geq 2 \cdot \mathbb{E}[X]] \leq 1/2.$$

$\therefore$ with prob. $\geq \frac{1}{2}$ my total cost is $\leq 2 \cdot \mathbb{E}[\text{cost}]$ $= 2 \cdot t \cdot \text{OPT}(\text{LP})$

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Given z optimal for the LP, our randomized rounding outputs, with probability ≥ 0.45 a feasible solution to set cover with $\leq 2 \cdot (\ln(|U|) + 3) \cdot OPT(ILP)$ sets

solution to the ILP

low cost

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3. Markov $\Rightarrow$ probability that our solution has weight $\geq 2 \cdot \omega$ is $\leq 1/2$
4. Union bound, with probability ≤ 0.55 either run for more than t times, or our solution has weight $\geq 2\omega$
5. Thus, with probability ≥ 0.45 we stop at t iterations **and** construct solution to set cover with cost $\leq 2t \cdot OPT(ILP)$

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   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If have *fractional values*, *rounding procedure*

      Randomized Rounding algorithm, with probability ≥ 0.45 we get

$$cost(\text{rounded solution}) \leq 2 \cdot (\ln(|U|) + 3) \cdot OPT(ILP)$$

$O(\log(|U|))$-approximation algorithm

Conclusion

- Integer Linear programming - very general, and pervasive in (combinatorial) algorithmic life
- ILP NP-hard
- Rounding for the rescue!
- Solve LP and round the solution
  - Deterministic rounding when solutions are nice
  - Randomized rounding when things a bit more complicated

Acknowledgement

- Lecture based largely on:
  - Lectures 7-8 of Luca's Optimization class
- See Luca's vertex cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture07.pdf
- See Luca's set cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture08.pdf