Lecture 12: Linear Programming and Duality Theorems

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Overview

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Part II
  - Game Theory
  - Learning Theory - Boosting
- Conclusion
- Acknowledgements

Mathematical Programming

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$$\begin{array}{rl} \text{minimize} & \underbrace{f(x)}_{\text{objective function}} \\ \text{subject to} & \left.\begin{array}{l} g_1(x) \leq 0 \\ \quad\vdots \\ g_m(x) \leq 0 \end{array}\right\} \text{Constraints} \\ & x \in \mathbb{R}^n \end{array}$$

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- Very general family of problems.

For instance: NP-hard when g_i's are quadratic polynomials!

Question: how much harder can it get?
(much harder!)

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- Very general family of problems.
- Special case is when all functions $f, g_1, \ldots, g_m$ are *linear* functions (called *Linear Programming* - LP for short)
- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x) = \underbrace{c_1}_{\in \mathbb{R}} \cdot x_1 + \ldots + \underbrace{c_n}_{\in \mathbb{R}} \cdot x_n = c^T x$$

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad c^T = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

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Linear Programming deals with problems of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & A_1^T x \leq 0 \\ & \vdots \\ & A_m^T x \leq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

$A_i \in \mathbb{R}^n$

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$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq 0 \\ & x \in \mathbb{R}^n \end{array}$$

$$A \in \mathbb{R}^{m \times n}$$

$$\begin{pmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_n^T \end{pmatrix}$$

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Linear Programming deals with problems of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

We can *always* represent LPs in *standard form*:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$
$b \in \mathbb{R}^m$

Practice problem: show that we can always represent a LP in standard form.

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!

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  - n companies, stock of company i costs $c_i \in \mathbb{R}$
  - company i has expected profit $p_i \in \mathbb{R}$
  - our budget is $B \in \mathbb{R}$

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total profit

$$\begin{array}{rl} \text{maximize} & \boxed{p_1 \cdot x_1 + \cdots + p_n \cdot x_n} \\ \text{subject to} & c_1 \cdot x_1 + \cdots + c_n \cdot x_n \leq B \\ & x \geq 0 \end{array}$$

cannot spend more than B to create our portfolio

Variables: x_i amount of stock i that you want to have

$x_i \cdot p_i$ profit from stock i

Why should I care?

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$$\begin{aligned} \text{maximize} \quad & p_1 \cdot x_1 + \cdots + p_n \cdot x_n \\ \text{subject to} \quad & c_1 \cdot x_1 + \cdots + c_n \cdot x_n \leq B \\ & x \geq 0 \end{aligned}$$

- Other problems, such as *data fitting, linear classification* can be modelled as linear programs.

Important Questions

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

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1. When is a Linear Program *feasible*?
   - Is there a solution to the constraints at all?

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3. Can we characterize *optimality*?
   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?

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4. How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs? interior point methods & ellipsoid

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Fundamental Theorem of Linear Inequalities

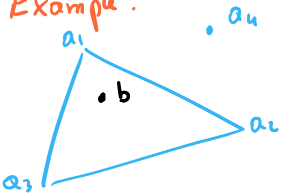
Theorem (Farkas (1894, 1898), Minkowski (1896))

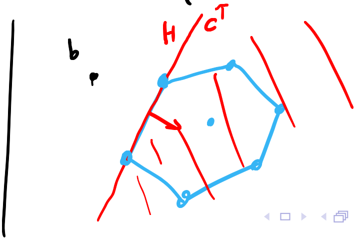
Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, and $t := \operatorname{rank}\{a_1, \ldots, a_m, b\}$. Then either

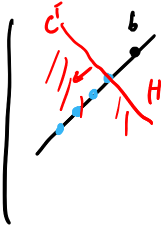
1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or*
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
   - *H contains $t - 1$ linearly independent vectors from $a_1, \ldots, a_m$*

$\operatorname{rank}\{v_1, \ldots, v_x\}$ = max number of linearly independent vectors in $\{v_1, \ldots, v_x\}$

Example:







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Remark

The hyperplane H above is known as the *separating hyperplane*.

Farkas' Lemma

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$* (our LP feasible)
2. *$y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$*

① ⇒ ② $\quad y^T A \geq 0 \quad Ax = b,\ x \geq 0 \quad \underbrace{\underbrace{y^T A}_{\geq 0}\underbrace{x}_{\geq 0}}_{\geq 0} = y^T b \Rightarrow y^T b \geq 0.$

② ⇒ ① (not ① ⇒ not ②)

$A_1, \dots, A_n$ columns of A, $A_1, \dots, A_n, b \in \mathbb{R}^m$

$\nexists\, x \geq 0$ s.t. $\sum_{i=1}^{n} x_i A_i = b$

(i.e. b is not nonnegative combination of $A_1, \dots, A_n$)

Fundamental thm of linear inequalities ⇒ then must have a separating hyperplane H_y s.t. $y^T A_i \geq 0\ \forall i$ and $\boxed{y^T b < 0}$ ⇒ not ②

$y^T A_i \geq 0\ \forall i \iff \boxed{y^T A \geq 0}$

Farkas' Lemma

Lemma (Farkas Lemma)

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1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
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Equivalent formulation

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$*

Linear Programming Duality

Consider our linear program:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\begin{aligned} y^T A \leq c^T &\Rightarrow y^T Ax \leq c^T x \\ &\Rightarrow y^T b \leq c^T x \end{aligned}$$

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$$\begin{aligned} y^T A \leq c^T &\Rightarrow y^T A x \leq c^T x \\ &\Rightarrow y^T b \leq c^T x \end{aligned}$$

- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

Linear Programming Duality

Consider the following linear programs:

Primal LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual LP

best

lower bound

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & \underbrace{y^T A \leq c^T}_{\text{constraint}} \end{array}$$

dual LP lower bounds the objective function of the primal LP.

Linear Programming Duality

Consider the following linear programs:

Primal LP	*Dual LP*
minimize $c^T x$	maximize $y^T b$
subject to $Ax = b$	subject to $y^T A \leq c^T$
$x \geq 0$	

- From previous slide

$$y^T A \leq c^T \Rightarrow y^T b \text{ is a lower bound on value of Primal}$$

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- Thus, the optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!

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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

Primal LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual LP

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Remarks on Duality

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Remarks on Duality

Primal LP		*Dual LP*	
$\alpha^* :=$ minimize	$c^T x$	$\beta^* =$ maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!
- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.

Remarks on Duality

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- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!
- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.
    - We showed that when both primal and dual are feasible, we have

$$\text{max dual } = \beta^* \leq \alpha^* = \text{ min of primal}$$

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  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)

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  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)

Remarks on Duality

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
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- **Practice problem:** show that dual of the dual LP is the primal LP!

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  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)
  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)
- **Practice problem:** show that dual of the dual LP is the primal LP!
- When is the above inequality tight?

Strong Duality

Primal LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{maximize} \quad & y^T b \\ \text{subject to} \quad & y^T A \leq c^T \end{aligned}$$

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Strong Duality

Primal LP	
minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

Dual LP	
maximize	$y^T b$
subject to	$y^T A \leq c^T$

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\text{max dual} = \boxed{\beta^* = \alpha^*} = \text{min of primal.}$$

always equality

Proof of Strong Duality

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If primal LP or dual LP is feasible, then

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1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.

Proof of Strong Duality

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If primal LP or dual LP is feasible, then

$$\text{max dual } = \beta^* = \alpha^* = \text{ min of primal.}$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \underbrace{\begin{pmatrix} A \\ -c^T \end{pmatrix}}_{\mathbb{R}^{(m+1)\times n}}$ and $v(\varepsilon) = \begin{pmatrix} b \\ \underbrace{-\alpha^* + \varepsilon}_{-(\alpha^* - \varepsilon)} \end{pmatrix}$ $\quad \varepsilon \geq 0$

$\alpha^* - \varepsilon \leq \alpha^* = \text{min primal}$

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal}.$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$
3. Apply Farkas' lemma on $Bx = v(0)$ and $x \geq 0$. This system has a solution, so we get:

x^*

$$(y^T \; z) B \leq 0 \Rightarrow (y^T \; z)\begin{pmatrix} b \\ -\alpha^* \end{pmatrix} \leq 0 \Leftrightarrow y^T b - z\alpha^* \leq 0$$

$$y^T A - zc^T \leq 0 \Rightarrow y^T b - z\alpha^* \leq 0$$

$$\boxed{y^T A \leq zc^T \Rightarrow y^T b \leq z\alpha^*}$$

if $z = 0$ then we have $y^T A \leq 0 \Rightarrow y^T b \leq 0$.

Proof of Strong Duality

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If primal LP or dual LP is feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal.}$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$
3. Apply Farkas' lemma on $Bx = v(0)$ and $x \geq 0$. This system has a solution, so we get:
4. Now, if $\varepsilon > 0$, applying Farkas' lemma on system $Bx = v(\varepsilon)$ and $x \geq 0$ we get: $\underset{x \geq 0}{Bx = v(\varepsilon)}$ has no solution (by optimality of α^*)

Farkas lemma: there is $(y^T \; z) \in \mathbb{R}^{m+1}$ s.t. $y^T A \leq c^T z$ and $y^T b > z(\alpha^* - \varepsilon)$

by previous slide $z \neq 0$ (can assume that $z > 0$)

$\Rightarrow \exists\, y_\varepsilon$ s.t. $y_\varepsilon^T b > \alpha^* - \varepsilon$ $(\forall \varepsilon)$

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

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4. Now, if $\varepsilon > 0$, applying Farkas' lemma on system $Bx = v(\varepsilon)$ and $x \geq 0$ we get:
5. Thus, for any $\varepsilon > 0$ there is $y \in \mathbb{R}^m$ such that $y^T A \leq c^T$ and $\beta^* \geq y^T b > \alpha^* - \varepsilon$.

y feasible solution

$\Rightarrow \boxed{\beta^* = \alpha^*}$

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \leq b$$

have at least one solution, and suppose that inequality

$$c^T x \leq \delta$$

holds whenever x *satisfies* $Ax \leq b$. *Then, for some* $\delta' \leq \delta$ *the linear inequality*

$$c^T x \leq \delta'$$

is a non-negative linear combination of the inequalities of $Ax \leq b$.

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$$c^T x \leq \delta'$$

is a non-negative linear combination of the inequalities of $Ax \leq b$.

Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Two-player games

Setup:

- Two players (Alice and Bob)
- Each player has a (finite) set of strategies $S_A = \{1, \ldots, m\}$ and $S_B = \{1, \ldots, n\}$

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Bob

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Alice

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Bob

Alice

	Football	Opera
Football	$\boxed{(2,1)}$	(0,0)
Opera	(0,0)	$\boxed{(1,2)}$

Table: Battle of the sexes payoff matrices

Nash Equilibrium

Assuming players are rational, i.e. want to maximize their payoffs, we have:

Definition (Nash Equilibrium)

A strategy profile (i, j) is called a Nash equilibrium if the strategy played by each player is optimal, *given the strategy of the other player*. That is:

1. $A_{ij} \geq A_{kj}$ for all $k \in S_A$
2. $B_{ij} \geq B_{i\ell}$ for all $\ell \in S_B$

1. if Alice knew Bob playing j
then Alice has no incentive to not play i

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Table: Battle of the sexes payoff matrices

Bob

Alice

	Silent	Snitch
Silent	(-1,-1)	(-10,0)
Snitch	(0,-10)	$\boxed{(-5,-5)}$

Nash equilibrium

Table: Prisoner's dilemma

Mixed Strategies

Definition (Mixed Strategy)

A mixed strategy is a probability distribution over a set of pure strategies S. If Alice's strategies are $S_A = \{1, \ldots, n\}$, her mixed strategies are:

$$\Delta_A := \{x \in \mathbb{R}^n \mid \underbrace{x \geq 0 \text{ and } \|x\|_1 = 1}_{x \text{ is a probability distribution}}\}$$

$x_i \leftarrow \Pr[\text{Alice plays } i]$

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$$v_A(x, y) = \sum_{(i,j) \in S_A \times S_B} A_{ij} \underbrace{x_i y_j} = x^T A y$$

$$v_B(x, y) = \sum_{(i,j) \in S_A \times S_B} B_{ij} \underbrace{x_i y_j} = x^T B y$$

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Goalie

'Player

	Jump left	Jump right
kick left	(-1,1)	(1,-1)
kick right	(1,-1)	(-1,1)

Table: Penalty Kick

no pure NE (player always has incentive to deviate)

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$\boxed{A+B=0}$

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- No pure Nash Equilibrium! Practice problem!

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Table: Penalty Kick

- *Zero-Sum Game*: payoff matrices satisfy $A = -B$
- No pure Nash Equilibrium!
- One mixed Nash equilibrium: $x = y = (1/2, 1/2)$

Practice problem!

Von Neumann's Minimax Theorem

Theorem

In a zero-sum game, for any payoff matrix $A \in \mathbb{R}^{m \times n}$:

$$\alpha \quad \underbrace{\max_{x \in \Delta_A}}_{\text{Alice wants to max her payoff}} \min_{y \in \Delta_B} \underbrace{x^T A y}_{\text{Alice's payoff}} = \underbrace{\min_{y \in \Delta_B} \max_{x \in \Delta_A} x^T A y}_{\text{Alice's payoff once Bob tells his strategy}} \quad \beta$$

Choose x so that whatever Bob plays Alice gets still α

LHS: Alice picks first her strategy

RHS: Bob picks first his strategy

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For given $x \in \Delta_A$:

$$\min_{y \in \Delta_B} x^T A y = \min_{j \in S_B} (x^T A)_j$$

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Left hand side can be written as

$$\begin{aligned}
\max \quad & s \\
\text{s.t.} \quad & s \leq (x^T A)_j \quad \text{for } j \in S_B \\
& \sum_{i \in S_A} x_i = 1 \\
& x \geq 0
\end{aligned}$$

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In a zero-sum game, for any payoff matrix $A \in \mathbb{R}^{m\times n}$:

$$\max_{x\in\Delta_A} \min_{y\in\Delta_B} x^T Ay = \min_{y\in\Delta_B} \max_{x\in\Delta_A} x^T Ay$$

For given $x \in \Delta_A$:

$$\min_{y\in\Delta_B} x^T Ay = \min_{j\in S_B} (x^T A)_j$$

For given $y \in \Delta_B$:

$$\max_{x\in\Delta_A} x^T Ay = \max_{i\in S_A} (Ay)_i$$

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pure strategy

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any pure strategy for Bob

over all Alice's strategy

For given $y \in \Delta_B$:

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Right hand side can be written as

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These two LPs are a primal-dual pair! ($\Rightarrow$ strong duality)

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$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & t \geq (Ay)_i \quad \text{for } i \in S_A \\ & \sum_{j \in S_B} y_j = 1 \\ & y \geq 0 \end{aligned}$$

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Part II
  - Game Theory
  - Learning Theory - Boosting
- Conclusion
- Acknowledgements

Learning Theory

Consider classification problem over $\mathcal{X}$:

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unknown
distribution
over elements

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- Data is sampled from unknown distribution $q \in \Delta_{\mathcal{X}}$
- *Weak learning assumption*:

For any distribution $q \in \Delta_{\mathcal{X}}$, there is a hypothesis $h \in \mathcal{H}$ which is wrong less than half the time.

$$\underbrace{\exists \gamma > 0}_{\text{gap}},\ \underbrace{\forall q \in \Delta_{\mathcal{X}}}_{\text{any dist.}},\ \underbrace{\exists h \in \mathcal{H}}_{\text{there is hypothesis}},\quad \underbrace{\Pr_{x\sim q}[h(x) \neq c(x)]}_{\text{Pr } h \text{ being wrong}} \leq \underbrace{\frac{1-\gamma}{2}}_{\text{less than } 1/2 \text{ the time (by gap } \gamma)}$$

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  For any distribution $q \in \Delta_{\mathcal{X}}$, there is a hypothesis $h \in \mathcal{H}$ which is wrong less than half the time.

  $$\exists \gamma > 0,\ \forall q \in \Delta_{\mathcal{X}},\ \exists h \in \mathcal{H}, \quad \Pr_{x \sim q}[h(x) \neq c(x)] \leq \frac{1-\gamma}{2}$$

- Surprisingly, weak learning assumption implies something much stronger: it is possible to *combine* classifiers in $\mathcal{H}$ to construct a *classifier* that is *always right* (known as *strong learning*).

Boosting

on set of hypotheses

Theorem

Let $\mathcal{H}$ be a set of hypotheses satisfying *weak learning assumption*. *Then there is distribution $p \in \Delta_{\mathcal{H}}$ such that the* *weighed majority classifier*

$$c_p(x) := \begin{cases} 1, & \text{if } \sum_{h \in \mathcal{H}} p_h \cdot h(x) \geq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

is always correct. That is, $c_p(x) = c(x)$ for all $x \in \mathcal{X}$

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- Let $M \in \{-1, 1\}^{m \times n}$, where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

$$M_{ij} = \begin{cases} +1, & \text{if classifier } h_j \text{ wrong on } x_i \\ -1, & \text{otherwise} \end{cases}$$

payoff matrix

classifiers

examples

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- Weak learning:

$q \in \Delta_X$

$$\sum_{1 \leq i \leq m} q_i \cdot \underbrace{\delta_{h_j(x_i) \neq c(x_i)}}_{} \leq \frac{1-\gamma}{2}$$

LHS: prob. that h is wrong

$\begin{cases} 1 & \text{if } h \text{ wrong on } x_i \\ 0 & \text{o.w.} \end{cases}$

Boosting - Proof

Let $M \in \{-1, 1\}^{m \times n}$,
where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

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Weak learning:

$$\sum_{1 \leq i \leq m} q_j \cdot \delta_{h_j(x_i) \neq c(x_i)} \leq \frac{1-\gamma}{2}$$

- Note that $M_{ij} = 2 \cdot \delta_{h_j(x_i) \neq c(x_i)} - 1$

$$\boxed{q^T M e_j \leq -\gamma} \quad \Rightarrow \quad q^T M p \leq -\gamma$$

for any $p \in \Delta_{\mathcal{H}}$.

$$\sum q_i \; 2 \cdot \delta_{i,j} \; - \underbrace{\sum q_i}_{1} \; \leq -\gamma$$

$$q^T M e_j = \sum q_i (2\delta_{ij} - 1) \leq -\gamma$$

Boosting - Proof

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where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

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Weak learning:

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$$q^T M e_j \le -\gamma \quad \Rightarrow \quad \underbrace{q^T M p}_{\text{value of our game}} \le -\gamma$$

  for any $p \in \Delta_{\mathcal{H}}$.
- By minimax, we have:

$$\max_{q \in \Delta_{\mathcal{X}}} \min_{p \in \Delta_{\mathcal{H}}} q^T M p = \underbrace{\min_{p \in \Delta_{\mathcal{H}}}}_{\text{classifiers}} \max_{q \in \Delta_{\mathcal{X}}} q^T M p \boxed{\le -\gamma}$$

Boosting - Proof

$\sum p_j \cdot h_j(x) \geq \frac{1+\gamma}{2} \Rightarrow c_p(x)=1$

Let $M \in \{-1,1\}^{m \times n}$,
where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

$$M_{ij} = \begin{cases} +1, & \text{if } h_j \text{ wrong on } x_i \\ -1, & \text{otherwise} \end{cases}$$

Weak learning:

$$\sum_{1 \leq i \leq n} q_j \cdot \delta_{h_j(x_i) \neq c(x_i)} \leq \frac{1-\gamma}{2}$$

$c(x)=1$

$\sum p_j \, \delta_{h_j(x)=0} \leq \frac{1-\gamma}{2}$

- Note that $M_{ij} = 2 \cdot \delta_{h_j(x_i) \neq c(x_i)} - 1$

$$q^T M e_j \leq -\gamma \quad \Rightarrow \quad q^T M p \leq -\gamma$$

for any $p \in \Delta_{\mathcal{H}}$.

$\sum p_j \, \delta_{h_j \text{ wrong}} \leq \frac{1-\gamma}{2}$

- By minimax, we have:

$$\max_{q \in \Delta_{\mathcal{X}}} \min_{p \in \Delta_{\mathcal{H}}} q^T M p = \boxed{\min_{p \in \Delta_{\mathcal{H}}} \max_{q \in \Delta_{\mathcal{X}}} q^T M p \leq -\gamma}$$

- In particular, right hand side implies weighted classifier *always* correct.

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Let $M \in \{-1, 1\}^{m \times n}$,
where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

$$M_{ij} = \begin{cases} +1, & \text{if } h_j \text{ wrong on } x_i \\ -1, & \text{otherwise} \end{cases}$$

Weak learning:

$$\sum_{1 \le i \le n} q_j \cdot \delta_{h_j(x_i) \neq c(x_i)} \le \frac{1-\gamma}{2}$$

- By minimax, we have:

$$\max_{q \in \Delta_{\mathcal{X}}} \min_{p \in \Delta_{\mathcal{H}}} q^T M p = \min_{p \in \Delta_{\mathcal{H}}} \max_{q \in \Delta_{\mathcal{X}}} q^T M p \le -\gamma$$

- In particular, right hand side implies weighted classifier *always* correct.

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- Linear Programming and Duality - fundamental concepts, lots of applications!
  - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
  - Applications in Game Theory (minimax theorem)
  - Applications in Learning Theory (boosting)
  - many more

Acknowledgement

- Lecture based largely on:
  - Lectures 3-6 of Yarom Singer's Advanced Optimization class
  - [Schrijver 1986, Chapter 7]
- See Yarom's notes at `https://people.seas.harvard.edu/~yaron/AM221-S16/schedule.html`

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