Lecture 10: Algebraic Techniques
Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

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- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)

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- many more...

Verifying String Equality

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can formalize using information theory

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Communication complexity setting, randomized algorithms, need to work with high probability.

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$(a_1, a_2, \ldots, a_n) \leftarrow a$ in base 2

$(b_1, b_2, \ldots, b_n) \leftarrow b$ in base 2

uniqueness of representation in base 2

$\Rightarrow \quad a = b \Leftrightarrow (a_1, \ldots, a_n) = (b_1, \ldots, b_n)$

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- what happens when they are different?

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- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)

equivalent to

$$M = F_p(a) - F_p(b) \stackrel{?}{=} 0$$

$F_p(a), F_p(b) \in \{0, \ldots, p-1\}$

$$M \in [-2p, 2p]$$

"M not too big"

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M "small" ⇒ not too many primes divide M

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  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$

$$|M| = |A_M| \cdot p_1 p_2 p_3 \cdots p_t > 1 \cdot 2^t$$

>0 >0

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- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.

$M = a - b \Rightarrow |M| \leq |a-b| \leq 2^n$

so $a - b \neq 0 \Rightarrow$ at most n primes p will divide $a - b$

$\therefore$ (at most n bad primes)

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- Choosing p among the ~~first~~ primes ≤ $tn \log(tn)$ ~~primes~~ integers we have that

$$\Pr[F_p(a) \equiv F_p(b)] \leq \frac{n}{tn \log tn / \log(tn \log tn)} = \tilde{O}(1/t)$$

$O(1/t \cdot \log n)$

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- Number of bits sent is $\tilde{O}(\log t + \log n)$. Choosing $t = n$ solves it.

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$$(a_1, \dots, a_n) = (b_1, \dots, b_n)$$

$$\Updownarrow$$

$$P_a(x) = \sum_{i=1}^{n} a_i x^i = P_b = \sum_{i=1}^{n} b_i x^i$$

Practice problem: give a different algorithm for the string equality problem using PIT

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$$\prod_{i=1}^{n} (x^{2^i}+1) = \sum_{\substack{(a_1,\dots,a_n) \\ \in \{0,1\}^n}} x^{\sum a_i 2^i}$$

"implicit representation small" (good)

2^n coefficients (bad)

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  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)

even in this setting let's see how to beat the "naive" algorithm

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- Multiplication of two polynomials of degree n: $O(n \log n)$ by FFT
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Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

Polynomial Identity Testing

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Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

"Proof": $\overline{\mathbb{F}}[x]$ is an Euclidean domain (have division with remainder)

α is root of $P(x)$

$$P(x) = (x-\alpha)\,Q(x)$$

$\deg(Q) = d-1$

induction.

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- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$

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- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.

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- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$

multiplying scalars!

Running time: evaluate $P_1(a)$ $P_2(a)$ $P_3(a)$ — $O(n)$ time

Checking $P_3(a) - P_1(a) \cdot P_2(a) = 0$ is $O(1)$ time.

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- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

← # roots of Q (upper bd)

↖ |S|

Polynomial Identity Testing

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$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

- Can amplify probability by running multiple times or by choosing larger set S.

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ be a field and $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be a nonzero polynomial of degree $\leq d$. Then for any set $S \subseteq \overline{\mathbb{F}}$, we have:

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid a_i \in S] \leq \frac{d}{|S|}$$

Upshot: lemma gives us a randomized algorithm for the polynomial identity testing problem!

Polynomial Identity Testing

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Let $\mathbb{F}$ be a field and $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be a nonzero *polynomial of degree $\leq d$. Then for any set $S \subseteq \overline{\mathbb{F}}$, we have:*

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid a_i \in S] \leq \frac{d}{|S|}$$

Proof by induction in number of variables.

for $n=1$ previous lemma.

Suppose lemma true for $n-1$ variables any degree.

$$P(x_1, \ldots, x_n) = \sum_{i=0}^{d} p_i(x_1, \ldots, x_{n-1})\, x_n^i$$ nonzero $\Rightarrow$ one of p_i's is nonzero. $\deg(p_i) \leq d-i$ because $\deg(p) \leq d$.

By induction $\Pr[p_i(a_1, \ldots, a_{n-1}) = 0] \leq \frac{d-i}{|S|}$

if $p_i(a_1, \ldots, a_{n-1}) \neq 0$ then $P(a_1, \ldots, a_{n-1}, x_n) \neq 0$ in $\mathbb{F}[x_n]$

$n=1$ case!

let i be largest index s.t. $p_i(\bar{x}) \neq 0$

$$\Pr[P(\overbrace{a_1, \dots, a_n}^{\bar{a}}) = 0] =$$

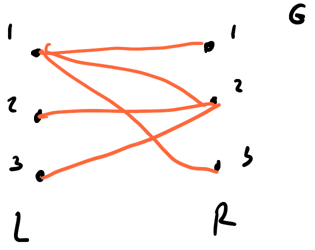
$$= \underbrace{\Pr[P(\bar{a}) = 0 \mid P_i(\bar{a}) = 0]}_{\leq 1} \cdot \underbrace{\Pr[P_i(\bar{a}) = 0]}_{\leq \frac{d-i}{|S|}}$$

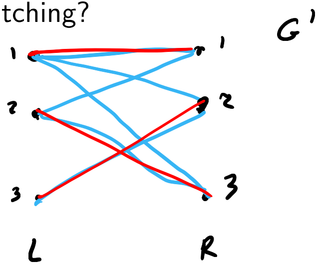
$$+ \underbrace{\Pr[P(\bar{a}) = 0 \mid P_i(\bar{a}) \neq 0]}_{\leq \frac{i}{|S|}} \cdot \underbrace{\Pr[P_i(\bar{a}) \neq 0]}_{\leq 1}$$

$$\leq 1 \cdot \frac{d-i}{|S|} + \frac{i}{|S|} \cdot 1 = \frac{d}{|S|} \quad \blacksquare$$

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?





L \ R	1	2	3
1	1	1	1
2	0	1	0
3	0	1	0

L \ R	1	2	3
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2	1	0	1
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matchings ⟺ permutations (S_n)

$\sigma(1) = 1$
$\sigma(2) = 3$
$\sigma(3) = 2$

σ corresponds to red matching!

---

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$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

Symbolic adjacency matrix of G

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-

$$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

sign(σ)

permutations of [n]

corresponds to matching (or zero) in G

---

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combinatorics

PIT (polynomial identity testing!)

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- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]
- Testing if G has a perfect matching is a *special case* of *Polynomial Identity Testing*!
- **Algorithm:** evaluate $\det(X)$ at a random value for the variables $y_{i,j}$.

$y_{ij} \leftarrow a_{ij} \quad a_{ij} \in [2n) \Rightarrow \det(A) \neq 0$ w.p. $\geq \frac{1}{2}$

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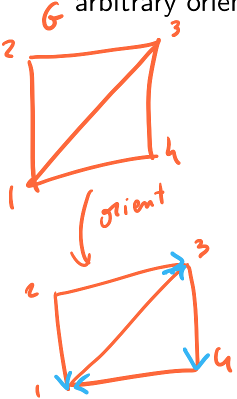

$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

T_G :

	1	2	3	4
1	0	$-x_{12}$	x_{13}	$-x_{14}$
2	x_{12}	0	x_{23}	0
3	$-x_{13}$	$-x_{23}$	0	x_{34}
4	x_{14}	0	$-x_{34}$	0

Skew-symmetric ($x_{ij} = -x_{ji}$)

General Matching

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Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Combinatorics PIT

Proof of Tutte's Theorem

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- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{2n}$.

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- Each vertex in H_σ has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.

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- If σ only has even cycles, then H_σ gives us a perfect matching (by taking every other edge of the graph H_σ, ignoring orientation)

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- Otherwise, for each $\sigma \in S_{2n}$ (that has odd cycle), there is another permutation $r(\sigma) \in S_{2n}$ that is obtained by reversing odd cycle of H_σ containing vertex with *minimum index*.

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- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)
- If T_G has a matching, say, $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$, then take permutation $\sigma = (1\ 2)(3\ 4)\cdots(2n-1\ 2n)$

$$(-1)^\sigma \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = (-1)^n \prod_{i=1}^{n} -x^2_{(2i-1)\sigma(2i-1)} = \prod_{i=1}^{n} x^2_{(2i-1)\sigma(2i-1)}.$$

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- We will see later in the course that we can

compute the determinant efficiently in parallel

(∴ evaluate it efficiently in parallel)

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

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need to isolate a matching without knowing it!

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- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order

in matchings
set random edge weights
and sort matchings by total weight

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- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$

$w(S_1) = 7$ $\quad w(S_2) = 13$ $\quad w(S_3) = 16$

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$w'(S_1) = 8 = w'(S_2)$ $\quad$ $w'(S_3) = 13$

no unique minimum

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Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is *only one* set of minimum weight.

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A and $B \setminus \{v\}$ do not depend on v !

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10. Probability that this happens is $\leq 1/2$. (step 8)

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- Cryptography
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- many more...

Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 7]
  - [Korte & Vygen 2012, Chapter 10].
- See Lap Chi's notes at
  `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf`

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