Lecture 8: Graph Sparsification

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Overview

- Introduction
  - Why Sparsify?
  - Warm-up Problem
- Main Problem
  - Graph Sparsification
- Acknowledgements

Why do we sparsify?

Often times graph algorithms for graphs $G(V, E)$ have runtimes which depend on $|E|$. If the graph is dense, i.e. $\underbrace{|E| = \omega(n^{1+c})}_{\text{super linear}}$ then this may be *too slow*.

We want graph that has nearly-linear number of edges $\underbrace{O(n \cdot \text{poly} \log n)}_{n \log^c n}$

- Settle for *approximate answers*

Algorithms will be randomized

$c > 0$
constant

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- Settle for *approximate answers*
- Used as primitives in many other algorithms (for instance, max-flow, sparsest cut, etc.)
- Applications in network connectivity

Graph Cuts

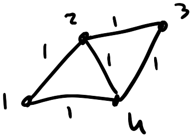
Definition (Graph Cut)

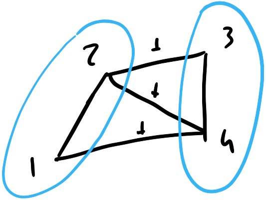
If $G(V, E, w)$ is a weighted graph, a *cut* is a partition of the vertices into two non-empty sets $V = S \sqcup \overline{S}$. The *value* of a cut is the quantity

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w_e.$$

$w : E \to \mathbb{R}_{\geq 0}$

$S = \{1, 2\}$ $\quad \overline{S} = \{3, 4\}$



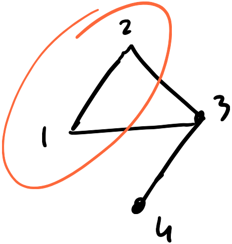


$w(S, \overline{S}) = 3$

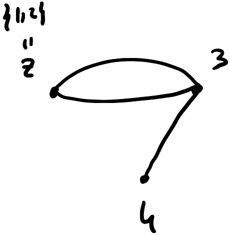
Contraction of Edges

Definition (Edge Contraction)

Let $G(V, E)$ be a graph. If $e = \{u, v\} \in E$ is an edge of G, then the *contraction* of e is a new graph $H(V \cup \{z\} \setminus \{u, v\}, F)$ where we replace the vertices u, v by *one* vertex z, and any edge $\{u, x\} =: f \in E \setminus \{e\}$ is replaced by $\{z, x\} \in F$. $\{v, x\}$

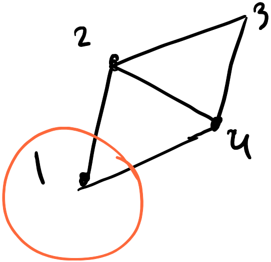


$e = \{1, 2\}$
contract



Randomized Minimum Cut

- **Input:** undirected unweighted connected graph $G(V, E)$
- **Output:** minimum cut $(S, \overline{S})$, with high probability



$S = \{1\}$
$S = \{3\}$
minimum cuts

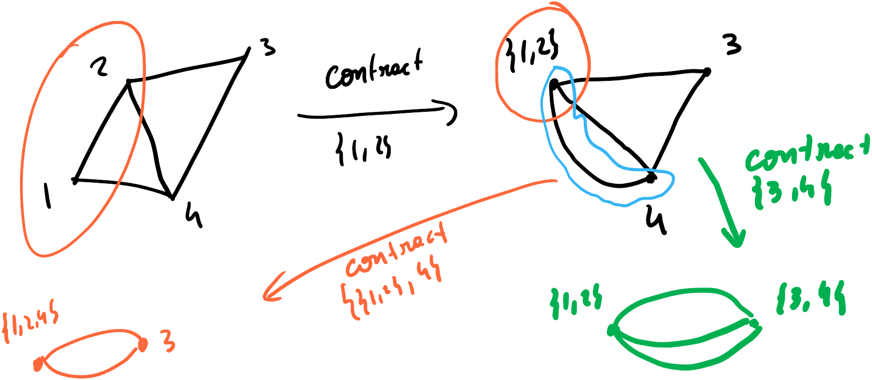
$S = \{1, 2\}$
$S = \{2\}$
not min. cuts

Randomized Minimum Cut

- **Input:** undirected unweighted graph $G(V, E)$
- **Output:** minimum cut $(S, \overline{S})$, with high probability
- While there are more than 2 vertices in the graph:
  - Pick uniformly random edge and contract it

Randomized Minimum Cut

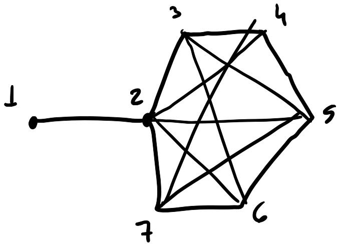
- **Input:** undirected unweighted graph $G(V, E)$
- **Output:** minimum cut $(S, \overline{S})$, with high probability
- While there are more than 2 vertices in the graph:
  - Pick uniformly random edge and contract it
- Output the two subsets encoded by the two remaining vertices.



Analysis

Why does this work?

Intuition: picking a random edge uniformly at random "favours" *small cuts* (i.e. preserves them) with higher probability.



$S = \{1\}$ only min cut

Analysis

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Remark

The value of the minimum cut only increases or stays the same after contraction.

$$G \xrightarrow[e]{\text{contract}} H$$

val. min-cut of $G \leq$ val. min-cut of H

Practice problem: prove this remark.

Analysis

Theorem (Karger)

The probability that the algorithm outputs a minimum cut is at least $2/n(n-1)$, *where* $n = |V|$.

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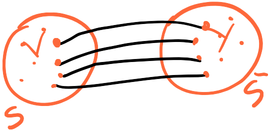
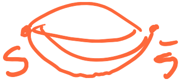
- Let $(S, \overline{S})$ be a minimum cut, and $k := |E(S, \overline{S})|$. If we never contract an edge from $E(S, \overline{S})$, the algorithm succeeds.



after all
$\longrightarrow$
contraction
(if no black edges contracted)



our min cut

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- Probability that an edge from $E(S, \overline{S})$ is contracted in the i^{th} iteration (conditioned on cut still alive)
  - Each vertex is a cut, so each vertex has degree $\geq k \Rightarrow$

$$\geq \frac{(n-i+1)\cdot k}{2} \quad \text{edges remain.}$$

at i^{th} iteration have $n-i+1$ vertices

(contracted $i-1$ times)

$$2|E_i| = \sum_{v \in H_i} \deg(v) \geq \sum_{v \in H_i} k = k\cdot(n-i+1)$$

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  - Contracting random edge, probability we kill cut $(S, \overline{S})$ is

$$= \underbrace{|E(S, \overline{S})|}_{\substack{\#\text{ edges} \\ \text{across cut} \\ = k}} \cdot \frac{1}{(\#\text{ edges})} \leq k \cdot \frac{2}{(n-i+1)\cdot k} = \frac{2}{n-i+1}$$

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  - $\Pr[(S, \overline{S}) \text{ survives}] \geq \underbrace{(1 - 2/n)}_{\text{survive } 1^{st}} \cdot \underbrace{(1 - \overset{2/n-1}{\cancel{3/n}})}_{\text{survive in } 2^{nd} \,|\, \text{survived } 1^{st}} \cdots (1 - 2/3) = 2/n(n-1)$

$$Pr[(S,\bar{S}) \text{ survives}] = Pr[(S,\bar{S}) \text{ survives } 1^{st} \text{ round}] \cdot$$

$$Pr[(S,\bar{S}) \; 2^{nd} \text{ rd} \mid (S,\bar{S}) \; 1^{st} \text{ rd}] \cdot \; \cdots$$

$$Pr[(S,\bar{S}) \text{ survives } i^{th} \text{ rd} \mid \text{still alive after rd } i-1]$$

$$= 1 - \underbrace{Pr[(S,\bar{S}) \text{ died at } i^{th}]}_{\leq \frac{2}{n-i+1}}$$

$$\geq 1 - \frac{2}{n-i+1}$$

Hmmmmm, this is not with high probability...

- To improve success probability, repeat this randomized procedure t times (for which t?)
- If we repeat for t times, failure probability is

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^t$$

Practice problem: prove this.

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- setting $t = 2n(n-1)$ then

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- **Running time:** One execution implemented in $O(n^2)$, so t executions in time $O(n^2 t) = O(n^4)$.
- For running time improvements, see [Motwani & Raghavan 2007, Chapter 10.2]

Combinatorial Application

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This is all good, but we haven't "sparsified" anything so far!

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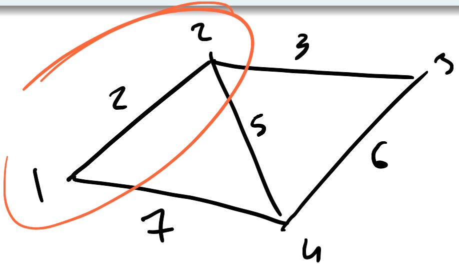
Graph Sparsification

Definition (Weight of a cut)

Let $G(V, E, w)$ be undirected weighted graph. For any cut $(S, \overline{S})$, let the weight of $(S, \overline{S})$ be

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w(e).$$



$w(\{1,2\},\{3,4\}) =$

$= 3 + 5 + 7 = 15$

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Definition (Sparse Graph)

We say that a graph $G(V, E)$ is *sparse* if $|E| = \tilde{O}(|V|)$.

$\tilde{O}(n) = O(n \log^c n)$

for some $c > 0$

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Question

How to make a graph sparse (nearly linear # edges) while approximating the value of every cut of a graph?

Graph Sparsification

- **Input:** graph $G(V, E, w_G)$, $\varepsilon > 0$.

$$n = |V|, \;\; m = |E|.$$

- **Output:** graph $H(V, F, w_H)$ such that *for every cut* $(S, \overline{S})$, we have

$$(1 - \varepsilon) \cdot w_G(S, \overline{S}) \leq w_H(S, \overline{S}) \leq (1 + \varepsilon) \cdot w_G(S, \overline{S})$$

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- *Assumption (for this class):* the input graph $G(V, E)$ is unweighted and has minimum cut value $\Omega(\log n)$ (i.e., a large-ish minimum cut)

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Algorithm:

- Let $p \in (0, 1)$ be a parameter.
- For each edge $e \in E(G)$, with probability p, make e an edge of H with weight $w_H(e) = 1/p$.

Graph Sparsification

Idea:

- Set p to get correct expected value for both $\#$ edges in H and the value of each cut $(S, \overline{S})$ in H.

not hard to prove

Graph Sparsification

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- Set p to get correct expected value for both $\#$ edges in H and the value of each cut $(S, \overline{S})$ in H.
- After that, need to prove concentration around expected values *for all cuts simultaneously*!

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Theorem ([Karger, 1993])

Let c be the value of the min-cut of G. Set

$$p = \frac{15 \ln n}{\varepsilon^2 \cdot c}.$$

Graph H given by algorithm from previous slide **approximates all cuts of** *G and has $O(p \cdot |E|)$ edges with probability $\geq 1 - 4/n$.*

Graph Sparsification

- Take a cut $(S, \overline{S})$. Suppose $k := w_G(S, \overline{S})$. Let
$$X_e = \begin{cases} 1, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$

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-

$$\underbrace{\mathbb{E}[|F|]}_{\text{expected \# edges in } H} = \sum_{e \in E} \mathbb{E}[X_e] = \sum_{e \in E} (p \cdot 1 + (1 - p) \cdot 0) = p \cdot |E|$$

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- $$\mathbb{E}[|F|] = \sum_{e \in E} \mathbb{E}[X_e] = \sum_{e \in E} (p \cdot 1 + (1 - p) \cdot 0) = p \cdot |E|$$
- Expected weight of cut $(S, \bar{S})$
$$\mathbb{E}[w_H(S, \overline{S})] = \sum_{e \in E(S, \overline{S})} \mathbb{E}[w_H(e)] = \sum_{e \in E(S, \overline{S})} (p \cdot \frac{1}{p} + (1 - p) \cdot 0)$$
$$= |E(S, \overline{S})| = k = w_G(S, \overline{S})$$

linearity of expectation

unweighted graph

Graph Sparsification - Concentration

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- $w_H(S, \overline{S})$ is a sum of independent random variables w_e

$$w_H(S, \bar{S}) = \sum_{e \in E(S, \bar{S})} w_e$$

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Important to use Chernoff

- $w_H(S, \overline{S})$ is a sum of independent random variables w_e
- Chernoff Bound: (why Chernoff if we not $\{0,1\}$-valued?)

$$\Pr[|w_H(S, \overline{S}) - k| \geq \varepsilon \cdot k] \leq 2 \exp\left(-\frac{\varepsilon^2 kp}{3}\right) = 2n^{-5k/c}$$

$w_e = \frac{1}{p} \cdot X_e$ $\qquad X_S = \sum_{e \in E(S,\bar{S})} X_e = p \cdot \sum w_e = p \cdot w_H(S, \bar{S})$

$$\Pr\left[|w_H(S,\bar{S}) - k| \geq \varepsilon k\right] = \underbrace{\Pr\left[|X_S - pk| \geq \varepsilon pk\right]}_{\text{use Chernoff here}}$$

$$\leq 2 \exp\left(-\frac{\varepsilon^2 pk}{3}\right)$$

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$p = \frac{15 \ln n}{\epsilon^2 \cdot c}$

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plugging in value of p

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- Note that $k \geq c$, as c is the weight of the minimum cut
- This is probability of *single cut* deviating from its mean... How can we handle the *exponentially many* cuts in the graph?

$(S, \bar{S})$ 2^{n-1} such pairs

Graph Sparsification - Concentration

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- **Observation:** probability that large cut violated is *much smaller*, and there are *not many small cuts*!

$k >> c$ then $n^{-5k/c}$ very small

learned from first part

Graph Sparsification - Concentration

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- So we can do a clever union bound!

Number of Cuts Lemma

$c \leftarrow$ value of minimum cut

Lemma (Number of small cuts)

The number of cuts with at most $\alpha \cdot c$ edges for $\alpha \geq 1$ is at most $n^{2\alpha}$.

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Practice problem: generalize our earlier proof on the # minimum cuts to this case.

Union Bound on # Cuts

$$\underbrace{\Pr[\text{some cut is violated}]}^{\text{want to show this is small}} \leq \sum_{S \subseteq V} \Pr[(S, \overline{S}) \text{ is violated}]$$

vanilla union bound

Union Bound on # Cuts

$$\Pr[\text{some cut is violated}] \le \sum_{S \subseteq V} \Pr[(S, \overline{S}) \text{ is violated}]$$

$$= \sum_{\alpha = 1,2,4,8,\ldots} \underbrace{\sum_{\substack{S \subseteq V \\ \alpha c \le |w_G(S,\overline{S})| \le 2 \cdot \alpha c}} \Pr[(S, \overline{S}) \text{ is violated}]}_{\text{Pr over all cuts which have value between } [\alpha c,\, 2\alpha c]}$$

grouping cuts based on their values

by cut lemma $\le n^{2(2\alpha)} = n^{4\alpha}$ such cuts

Union Bound on # Cuts

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$$\leq \sum_{\alpha = 1,2,4,8,\ldots} n^{4\alpha} \cdot \underbrace{\Pr[(S, \overline{S}) \text{ is violated} \mid \alpha c \leq |w_G(S, \overline{S})| \leq 2 \cdot \alpha c]}_{\text{conditioned on our cut having proper size}}$$

by Chernoff

$$\leq 2n^{-5k/c} \leq 2n^{-5\alpha c/c} = 2n^{-5\alpha}$$

Union Bound on # Cuts

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$$\leq \sum_{\alpha = 1,2,4,8,\ldots} n^{4\alpha} \cdot 2n^{-5\alpha c/c}$$

$$= \sum_{\alpha = 1,2,4,8,\ldots} n^{-\alpha} \leq 4/n$$

for all cuts simultaneously

Union Bound on # Cuts

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Another application of Chernoff gives us that H has the right number of edges $|F| \approx p \cdot |E|$ (i.e., sparse)

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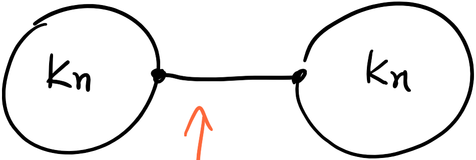
Counterexample:



cannot remove this edge!

if sample uniformly have that $p = O\left(\frac{1}{n^2}\right)$
prob that our bridge cut survives is $\leq \frac{1}{n^2}$

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- **[Benczur, Karger 1996]:** without minimum cut assumption, just sample non-uniformly in clever way!

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- Sample edge e with probability $p_e = \Theta\left(\frac{\log n}{\varepsilon^2 \cdot s_e}\right)$ and weight $1/p_e$.

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's Lecture 1 notes at
  https://cs.uwaterloo.ca/~lapchi/cs466/notes/L01.pdf
- See Lap Chi's Lecture 3 notes at
  https://cs.uwaterloo.ca/~lapchi/cs466/notes/L03.pdf
- See Mohsen's notes for the general Benczur-Karger algorithm
  https://people.inf.ethz.ch/gmohsen/AA18/Notes/S1.pdf.

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