Lecture 7: Sublinear Time Algorithms

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Overview

- Introduction
  - Why Sublinear Time Algorithms?
  - Warm-up Problem
- Main Problem
  - Number of Connected Components
- Acknowledgements

How do we handle big data? (part II)

Sometimes big data does not come to us (think streaming), but instead we *can query small pieces* of it.

Sometimes big data can also *change over time*, so we need a *robust* answer and/or be able to solve problem quickly multiple times.

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- Many more...

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Connects to *randomized algorithms*, *approximation algorithms*, *parallel algorithms*, *complexity theory*, *statistics*, *learning*

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What we *can't* do:

- Can't answer **for all** or **there exists** or **exactly** type statements

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What we *can* do:

- Can answer **for most** or **averages** or **approximate** type statements *with high probability*
  - are most individuals connected via friendships?
  - are most individuals connected by at most 6 degrees of separation?
  - approximately how many people are left handed?
  - is my program correct on most inputs

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Randomized & *Approximate* algorithms.

Sublinear Time Models of Computation

- Random Access Queries

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  - Can access any word of input in one step
  - How is input represented?

Sublinear Time Models of Computation

Sublinear time in Adjacency matrix model $o(N^2)$ time

Sublinear time in Adjacency list model $o(M+N)$ time

- Random Access Queries
  - Can access any word of input in one step
  - How is input represented?
    - Adjacency matrix
    - Adjacency list

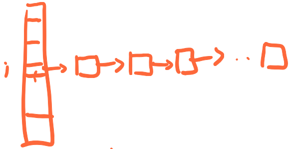
Adjacency matrix

$A \in \Sigma^{M \times N}$ $(M \leq N)$

query entry (i,j) of A

size input : $O(MN)$ $O(N^2)$

Adjacency list



M entries

N elements

size of input = $O(M+N)$

Sublinear Time Models of Computation

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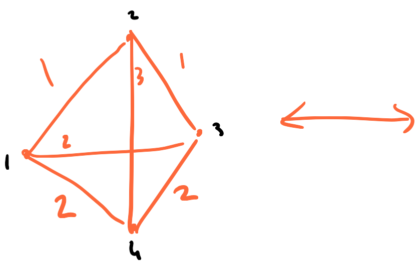
    - Location
    - many others...
- Samples
  - get samples from certain distribution/input at each step

Approximate Diameter of a Point Set

- **Input:** m points and a distance matrix D such that
  - $D_{ij} \leftarrow$ distance from i to j
  - D *symmetric* and satisfies *triangle inequality*

Input given in *adjacency matrix* representation



$D =$

	1	2	3	4
1	0	1	2	2
2	1	0	1	3
3	2	1	0	2
4	2	3	2	0

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$D_{24} = 3$ is our diameter

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- Let a, b be indices that *maximize* distance D_{ab}. Then D_{ab} is *diameter*
- **Output:** Indices k, ℓ such that

$$D_{k\ell} \geq D_{ab}/2$$

$D_{ab} = \max_{i,j} D_{ij}$

at least half of the diameter

2-multiplicative algorithm

Algorithm & Analysis

- Pick k arbitrarily

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- Pick ℓ to maximize D_{kl}
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Why does this work?

Algorithm & Analysis

- Pick k arbitrarily
- Pick ℓ to maximize $D_{k\ell}$
- Output indices k, ℓ

$D_{k\ell} \geq D_{kj}$

for any $j \in [m]$

Why does this work?

- Correctness

triangle inequality

D is symmetric

$$D_{ab} \leq D_{ak} + D_{kb} = D_{ka} + D_{kb}$$
$$\leq D_{k\ell} + D_{k\ell} = 2 \cdot D_{k\ell}$$

by property of choice of ℓ

Algorithm & Analysis

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$$D_{ab} \leq D_{ak} + D_{kb}$$
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- Running time: $O(m) = O(N^{1/2})$ = $o(N)$

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$$
\begin{aligned}
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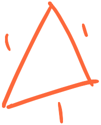
&\leq D_{k\ell} + D_{k\ell} = 2 \cdot D_{k\ell}
\end{aligned}
$$

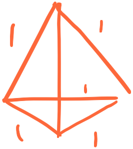
- Running time: $O(m) = O(N^{1/2})$

Is this the best we can do?

Lower Bound for Approximate Diameter

- Let D be following: distance matrix $D_{i,i} = 0, \; \forall i \in [m]$ and $D_{i,j} = 1$ otherwise





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$$D'_{ab} = D'_{ba} = 2 - \delta$$

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- Check that D' satisfies properties of a distance matrix (thus valid)
- **Practice problem:** prove that it would take $\Omega(N)$ time (i.e. number of queries) to decide if diameter is 1 or $2 - \delta$

Connected Components

How to approximate number of connected components of a graph:

Connected Components

How to approximate number of connected components of a graph:

- **Input:** graph $G(V, E)$ in *adjacency list* representation. $\epsilon > 0$.

$$n = |V|, \; m = |E|, \; N = m + n$$

- **Output:** if $C \leftarrow \#$ connected components of G, output with probability $\geq 3/4$, C' such that

$$|C' - C| \leq \epsilon n$$

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Lemma (# Connected Components)

Let $G(V, E)$ be a graph. For vertex $v \in V$, let $n_v \leftarrow \#$ vertices in connected component of v. Let C be number of connected components of G. Then:

$$C = \sum_{v \in V} \frac{1}{n_v}$$

$v \longmapsto n_v \triangleq$ # vertices in connected component of v

$\underbrace{}_{\Gamma}$

$$\sum_{v \in \Gamma} \frac{1}{n_v} = \frac{|\Gamma|}{n_v} = \frac{n_v}{n_v} = 1$$

$$C = \sum_{\Gamma \text{ connected component of } C} 1_\Gamma = \sum_{\Gamma} \sum_{v \in \Gamma} \frac{1}{n_v} = \sum_{v \in V} \frac{1}{n_v}$$

Connected Components

Naive attempt: sample small number of vertices from G, compute n_v and output average.

$v_1, \ldots, v_a$

$n_{v_1}, \ldots, n_{v_a}$

output: $\frac{n}{a} \sum_{i=1}^{a} \frac{1}{n_{v_i}}$

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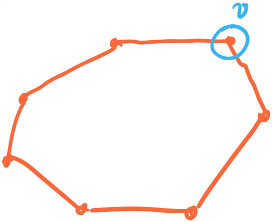
- **Problem:** just computing n_v may take *linear time* if graph is connected!



$n_v = |V|$

this is already linear time!

Connected Components

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- **Problem:** just computing n_v may take *linear time* if graph is connected!
- **Idea:** if n_v large, then $1/n_v$ small and we can drop it!

don't compute n_v exactly!

(just see if it is too large)

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Lemma (Estimating # components)

Let

$$n'_v = \min(n_v, 2/\epsilon)$$

proxy for size of connected component

Then

$$\left| \sum_{v \in V} \frac{1}{n_v} - \sum_{v \in V} \frac{1}{n'_v} \right| \le \frac{\epsilon n}{2}.$$

proxy for # connected components

$$\left| \sum_{v \in V} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right) \right| \leq \sum_{v \in V} \left| \frac{1}{n_v} - \frac{1}{n'_v} \right|$$

↓ triangle inequality

if $n_v \leq \frac{2}{\epsilon}$ then $n_v = n'_v \Rightarrow \frac{1}{n_v} - \frac{1}{n'_v} = 0$

else $n_v > \frac{2}{\epsilon} \Rightarrow 0 < \frac{1}{n_v} < \frac{\epsilon}{2} \Rightarrow \left| \frac{1}{n_v} - \underbrace{\frac{1}{n'_v}}_{\frac{\epsilon}{2}} \right| \leq \frac{\epsilon}{2}$

$$\leq \sum_{v \in V} \frac{\epsilon}{2} = \frac{\epsilon n}{2}$$

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How do we do this estimation?

Sample vertex v and run BFS starting at v, short-cutting if see $2/\epsilon$ vertices.

Connected Components - proof of lemma

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Problem: cannot compute $\sum_{v \in V} \frac{1}{n'_v}$ in sublinear time

Solution: let's sample a few vertices and scale our estimate

Algorithm

- Choose $s = \Theta(1/\epsilon^2)$ vertices $v_1, \ldots, v_s$ uniformly at random.

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- Return

$$C' = \frac{n}{s} \cdot \sum_{i=1}^{s} \frac{1}{n'_{v_i}}$$

normalized estimate

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- **Running Time:**

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  - Adding results takes $O(s) = O(1/\epsilon^2)$ time.
- Total running time $O(1/\epsilon^4)$.

Sublinear as it doesn't even depend on n

Algorithm - Correctness

To prove correctness we need to show that with probability $\geq 3/4$ we have

$$\left| \frac{n}{s} \cdot \sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \sum_{v \in V} \frac{1}{n_v} \right| \leq \epsilon n$$

estimate

connected components

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Dividing by n/s on both sides:

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By our previous lemma, and triangle inequality, enough to prove that w.h.p.

$$\left| \sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v} \right| \leq \frac{\epsilon s}{2}$$

Lemma and Triangle Inequality

Lemma (Estimating # components)

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Then

$$\left| \sum_{v \in V} \frac{1}{n_v} - \sum_{v \in V} \frac{1}{n'_v} \right| \leq \frac{\epsilon n}{2}.$$

$$\left| \sum_{i=1}^{s} \frac{1}{n_{v_i}} - \frac{s}{n} \sum_{v \in V} \frac{1}{n_v} \right| \overset{\text{triangle inequality}}{\leq} \left| \frac{s}{n} \sum_{v \in V} \frac{1}{n_v} - \frac{s}{n} \sum_{v \in V} \frac{1}{n'_v} \right| +$$

$$\left| \sum_{i=1}^{s} \frac{1}{n_{v_i}} - \frac{s}{n} \sum_{v \in S} \frac{1}{n'_v} \right| \overset{\text{by above lemma}}{\leq} \underbrace{\frac{s}{n} \cdot \frac{\epsilon n}{2}}_{\frac{\epsilon s}{2}} + \underbrace{\left| \sum_{i=1}^{s} \frac{1}{n_{v_i}} - \frac{s}{n} \sum_{v \in S} \frac{1}{n'_v} \right|}_{\leq \frac{\epsilon s}{2} \text{ w.h.p.}} \leq \epsilon s \text{ w.h.p.}$$

$$\Rightarrow |c' - c| \leq \epsilon n$$

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Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{N} X_i$. Then

$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^{N} (b_i - a_i)^2} \right)$$

Algorithm - Correctness

Want to show that with probability $\geq 3/4$:

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Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{N} X_i$. Then

$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^{N}(b_i - a_i)^2}\right)$$

Setting parameters of Hoeffing's theorem to our setting:

- $a_i = 0,\ b_i = 1,\ N = s$
- $X_i = 1/n'_v$ with probability $1/n$ (pick vertex uniformly at random)

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

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$$\mu := \mathbb{E}[X] = \sum_{i=1}^{s} \mathbb{E}[X_i] = s \cdot \sum_{v \in V} \frac{1}{n'_v} \cdot \frac{1}{n} = \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}$$

linearity of expectation

$\mathbb{E}[X_i]$

X_i : sample vertex v uniformly at random then compute $\frac{1}{n'_v}$

$$\mathbb{E}[X_i] = \sum_{v \in V} \Pr[\text{pick } v] \cdot \frac{1}{n'_v} = \sum_{v \in V} \frac{1}{n} \cdot \frac{1}{n'_v}$$

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

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Hoeffding with the parameters from previous slide and $\ell = \epsilon \cdot s/2$:

Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[0, 1]$, $X = \sum_{i=1}^{s} X_i$. Then

$$\Pr[|X - \mu| \geq \epsilon \cdot s/2] \leq 2 \cdot \exp\left(-\epsilon^2 s/2\right)$$

why need to take $s = \Theta\left(\frac{1}{\epsilon^2}\right)$

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

$$\mu := \mathbb{E}[X] = \sum_{i=1}^{s} \mathbb{E}[X_i] = s \cdot \sum_{v \in V} \frac{1}{n'_v} \cdot \frac{1}{n} = \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}$$

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Since $s = \Theta(1/\epsilon^2)$, the result follows by choosing $s = 8 \cdot (1/\epsilon^2)$

Acknowledgement

- Lecture based largely on Ronitt's notes.
- See Ronitt's notes at http://people.csail.mit.edu/ronitt/COURSE/F20/Handouts/scribe1.pdf
- See also her notes for approximate MST http://people.csail.mit.edu/ronitt/COURSE/F20/Handouts/scribe2.pdf
- List of open problems in sublinear algorithms https://sublinear.info/index.php?title=Main_Page

$X_i =$ Sample v at random, uniformly

Compute $\frac{1}{n_v'}$

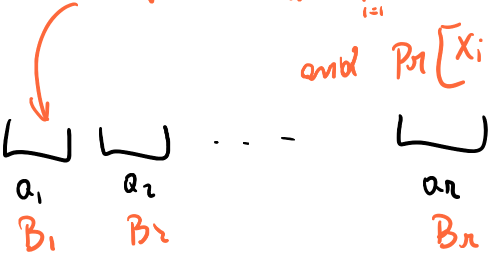
$$X_i \in \left\{ \frac{1}{n_v'} \right\} = S = \{a_1, a_2, \ldots, a_r\}$$

$B_i \cap B_j = \emptyset$

and $\bigcup_{i=1}^{r} B_i = V \Rightarrow \sum \frac{|B_i|}{n} = 1$

and $\Pr\left[X_i = \frac{1}{n_v'}\right] = \frac{|B_e|}{|V|} = \frac{|B_e|}{n}$

$v \quad \frac{1}{n_v'} = a_1$

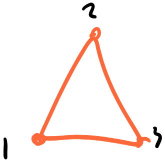


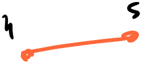
$$E[X_i] = \sum_{v \in V} Pr[X_i \text{ picked } v] \cdot \frac{1}{n'_v}$$

separate elements of a bucket

$$= \sum_{i=1}^{r} \frac{|B_i|}{n} \cdot \frac{1}{n'_v}$$

$$\sim Pr\left[X_i = \frac{1}{n'_v}\right]$$





$\{1, 2, 3\}$ $\{4, 5\}$

$\frac{1}{3}, \frac{1}{2}$

$$\frac{3}{5} \cdot \frac{1}{3} + \frac{2}{5} \cdot \frac{1}{2}$$