Lecture 5: Hashing

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Overview

- Introduction
  - Hash Functions
  - Why is hashing?
  - How to hash?
- Succinctness of Hash Functions
  - Coping with randomness
  - Universal Hashing
  - Hashing using 2-universal families
  - Perfect Hashing
- Acknowledgements

Computational Model

Before we talk about hash functions, we need to state our model of computation:

Definition (Word RAM model)

In the word RAM[a] model:

- all elements are integers that fit in a machine word of w bits
- Basic operations (comparison, arithmetic, bitwise) on such words take $\Theta(1)$ time
- We can also access *any* position in the array in $\Theta(1)$ time

[a] RAM stands for Random Access Model

What is hashing?

We want to store n elements (keys) from the set $U = \{0, 1, \ldots, m-1\}$, where $m >> n$, in a data structure that supports *insertions, deletions, search* "as efficiently as possible."

$m < 2^w$ $\quad\therefore$ can store each element of U in one word

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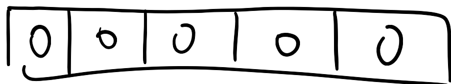
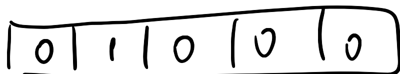


insert(1)



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- Insertion: $O(1)$, Deletion: $O(1)$, Search: $O(1)$
- Memory: $O(m \log(m))$ *(this is very bad!)*

if assume $\log(m) < w$

then $O(m)$ space

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- Memory: $O(m\log(m))$ *(this is very bad!)*

Want to also achieve optimal memory $O(n\log(m))$. For this we will use a technique called *hashing*.

or be as close as possible

- A *hash function* is a function $h : U \to [0, n-1]$, where $|U| = m >> n$.
- A *hash table* is a data structure that consists of:
  - a table T with n cells $[0, n-1]$, each cell storing $O(\log(m))$ bits
  - a hash function $h : U \to [0, n-1]$

From now on, we will define memory as *# of cells*.

Why is hashing useful?

- Designing efficient data structures (dictionaries) for searching
- Data streaming algorithms
- Derandomization

  When randomness needed only involves that pairs, or triples, or *small number of elements* "*look independent*."
- Cryptography

  Construct functions that look random to adversaries, but are easy for us to compute.
- Complexity Theory
- many more

Challenges in Hashing

Setup:

- Universe $U = \{0, \ldots, m-1\}$ of size $m >> n$ where n is the size of the range of our hash function $h : U \rightarrow [0, n-1]$
- Store $O(n)$ elements of U (keys) in hash table T (which has n cells)

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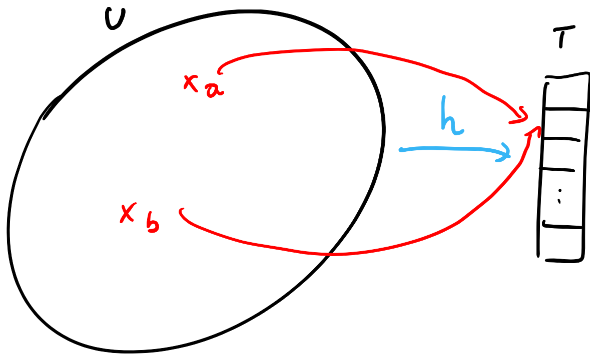
Ideally, want hash function to map *different keys* into *different locations*.

Definition (Collision)

A *collision* happens for hash function h with inputs $x, y \in U$ if

$$x \neq y \quad \text{and} \quad h(x) = h(y).$$

Pigeonhole principle $\Rightarrow$ impossible without knowing keys in advance.



$|U| > |T|$

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Will settle for: # collisions *small with high probability*.

Definition (perfect hashing): given a set $S \subset U$ and hash function $h : U \to [0, .., n-1]$

h is <u>perfect</u> for S if $\forall\, x, y \in S$, $x \neq y$

$$h(x) \neq h(y)$$

i.e. there are no collisions.

Our solution: family of hash functions

Construct *family* of hash functions $\mathcal{H}$ such that the *number of collisions* is **small** with **high probability**, when we pick hash function uniformly at random from the family $\mathcal{H}$.

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Simplest version to keep in mind:

$$\Pr_{h \in_R \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{\text{poly}(n)} \qquad \forall x \neq y \in U$$

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Assumptions:

- keys are independent from hash function we choose.
- we **do not** know keys in advance (even if we did, nontrivial problem!)

Question

Still could have collisions. How do we handle them?

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Natural to consider following approach:

From all functions $h : U \to [0, n-1]$, just pick one uniformly at random.

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This setting is same as our balls-and-bins setting!

$$h : [0, m-1] \to [0, n-1]$$

balls — bins

taking n labelled balls from U ⇔ randomly throwing (unlabeled) n balls into bins (+ h)

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- Expected number of keys in a location: 1
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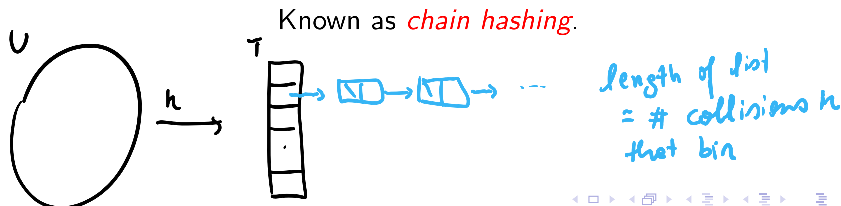
Known as *chain hashing*.



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Could also pick *two* random hash functions and use *power of two choices*.
Collision bound becomes $O(\log \log n)$.

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- Storing entire function $h : U \to [0, n-1]$ require $O(m \log n)$ bits (way too much space!)
- Even if we only stored the elements we saw, would require $O(n)$ time to evaluate $h(x)$ (need to decide if we had already computed it!)

$h(1) = 2 \quad h(3) = 1 \quad h(17) = 4 \ldots$

17 comes again need to check whether I already defined it

($O(n)$ time)

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How do we cope with the computational problem that arose with randomness?

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Ideally something that takes $O(\log m)$ time to compute (size of our input).

" word size "
$O(1)$

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poly(m) functions, as each function takes at most $O(\log m)$ bits to describe. Thus these are *succinct functions* (easy to describe and compute) which have *random-like* properties!

$$2^{O(\log m)} = 2^{c \cdot \log m} = m^c = \text{poly}(m)$$

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Part of *derandomization/pseudorandomness*: huge subfield in TCS!

k-wise independence

Weaker notion of independence.

"sacrifice some randomness to get better computation"

k-wise independence

Weaker notion of independence.

Definition (Full Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be (fully) independent if they satisfy

$$\Pr\left[\bigcap_{i=1}^{n} X_i = a_i\right] = \prod_{i=1}^{n} \Pr[X_i = a_i]$$

$\forall a_i$

↳ in range of X_i

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Definition (k-wise Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be k-wise independent if for any set $J \subset [n]$ such that $|J| \leq k$ they satisfy

→ small subsets of our random variables

$\forall a_i$ in range X_i

$$\Pr\left[\bigcap_{i \in J} X_i = a_i\right] = \prod_{i \in J} \Pr[X_i = a_i]$$

Pairwise independence

When $k = 2$, k-wise independence is called *pairwise independence*.

Example (XOR pairwise independence)

Given b uniformly distributed, independent random bits $Y_1, \ldots, Y_b$, we can generate $2^b - 1$ pairwise independent, uniform random variables as follows:

$$X_S := \bigoplus_{i \in S} Y_i \qquad S \subseteq [b] \setminus \emptyset$$

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- Why are they uniformly random?

$S = \{1, 2, 3\}$

$X_S = Y_1 \oplus Y_2 \oplus Y_3 \rightarrow \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2 \end{cases}$

$Y_1 \oplus Y_2 = a$, $a \in \{0, 1\}$; Y_3: $1/2 \to a$, $1/2 \to 1-a$

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$$\Pr[X_{S_1} = a_1 \text{ and } X_{S_2} = a_2] = \Pr[X_{S_1} = a_1] \cdot \Pr[X_{S_2} = a_2]$$

$S_1 \neq S_2 \quad \exists\, j \in S_2 \setminus S_1$

even if we set all variables in S_1

$X_{S_2} = Y_j \oplus [\quad]$ (not set; $\frac{1}{2} \to 0$, $\frac{1}{2} \to 1$)

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- Why are they uniformly random?
- Why are they pairwise independent?
- Are they also 3-wise independent?

$X_{\{1\}}$ $\quad X_{\{2\}}$ $\quad X_{\{1,2\}}$

0 $\quad 0$ $\quad \Rightarrow 0$ w.p. 1

Pairwise independence II

Example (Pairwise independence in $\mathbb{F}_p$)

Let p be a prime number. Given 2 uniformly random variables $Y_1, Y_2 \sim [0, \ldots, p-1]$, generate p pairwise independent random variables as follows:

$$X_i := Y_1 + i \cdot Y_2 \mod p \qquad i \in [0, p-1]$$

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (integers mod p)

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- Why are they uniformly random?

$X_1 = Y_1 + Y_2 \mod p$

$a \in \mathbb{F}_p \quad (x, a-x)$

have p pairs (Y_1, Y_2) s.t. $X_1(Y_1, Y_2) = a$

$\Pr[X_1 = a] = \frac{p}{p^2} = \frac{1}{p}$

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- Why are they uniformly random?
- Why are they pairwise independent?

$\Pr[X_i = a_i \wedge X_j = a_j] = \frac{1}{p^2}$

$Y_1 + i Y_2 = a_i \mod p$

$Y_1 + j Y_2 = a_j \mod p$

} system of 2 equations (linearly independent) in 2 variables

$(j-i) Y_2 = (a_j - a_i)$, $j - i \neq 0$

$\therefore$ we have exactly one solution

$\therefore \Pr[X_i = a_i \wedge X_j = a_j] = \frac{1}{p^2}$

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NO.

$X_0 = 0$
$X_1 = 0$
$X_2 = 1$

Prob. 0 of happening

(inconsistent system of equations)

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Can think of these random variables as picking a random line.

- If we only know one point of the line, the second point is still uniformly random. *pairwise independence*
- Two points determine the line. *not 3-wise independent*

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Universal Hash Functions

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Definition (Universal Hash Functions)

Let U be a universe with $|U| \geq n$. A family of hash functions $\mathcal{H} = \{h : U \to [0, n-1]\}$ is *k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$, we have

$$\Pr_{h \in_R \mathcal{H}} [h(u_1) = h(u_2) = \ldots = h(u_k)] \leq 1/n^{k-1}$$

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Definition (Strongly Universal Hash Functions)

$\mathcal{H} = \{h : U \to [0, n-1]\}$ is *strongly k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$ and *for any* values $y_1, \ldots, y_k \in [0, n-1]$, we have

$$\Pr_{h \in_R \mathcal{H}} [h(u_1) = y_1, \ldots, h(u_k) = y_k] \leq 1/n^k$$

Relation to k-wise independent random variables

What do the previous definitions have to do with random variables?

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Family $\mathcal{H}$ is *strongly k-universal* if the random variables $h(0), \ldots, h(|U| - 1)$ are *k-wise independent*.

Relation to k-wise independent random variables

What do the previous definitions have to do with random variables?

Family $\mathcal{H}$ is *strongly k-universal* if the random variables $h(0), \ldots, h(|U| - 1)$ are *k-wise independent*.

Can use random variables to construct universal hash functions!

Strongly 2-universal families of hash functions

Let p be a prime number, $U = [0, p-1]$.

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := a \cdot x + b \mod p \quad | \quad a, b \in [0, p-1]\}$$

is strongly 2-universal.

$h_{a,b}(i) = a \cdot i + b$ $h_{a,b}(j) = a \cdot j + b$

Same proof we did before.

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Proposition

Let $U = [0, p^k - 1] \equiv [0, p-1]^k$ *and* $\mathrm{a} = (a_0, \ldots, a_{k-1}) \in \mathbb{F}_p^k$

$\mathbb{F}_{p^k} \equiv \mathbb{F}_p^k$

$$\mathcal{H} = \{h_{\mathrm{a},b}(\mathrm{x}) := \langle \mathrm{a}, \mathrm{x} \rangle + b \mod p \mid \mathrm{a} \in U, b \in [0, p-1]\}$$

$\hookrightarrow a_0 \cdot x_0 + a_1 x_1 + \cdots + a_{k-1} \cdot x_{k-1}$

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is strongly 2*-universal.*

Proof similar to first proof (subsets)

(there $p=2$)

2-universal families of hash functions

$\mathbb{F}_p \to \mathbb{F}_p \to \mathbb{Z}/n\mathbb{Z}$ simple hash function

$\mathbb{F}_p = U$

What if my hash table size is not a prime?

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := ((a \cdot x + b) \bmod p) \bmod n \quad | \quad a, b \in [0, p-1], a \neq 0\}$$

$[0, p-1] = \mathbb{F}_p = U$

is 2-universal (but not strongly 2-universal).

Practice problem: prove the proposition above.

k-universal families of hash functions

Can we construct k-universal families of hash functions like this?

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- YES! Instead of constructing random lines (degree 1 polynomials), can construct random univariate polynomials of degree $k - 1$

$$h_{a,b}(x) = ax + b$$

$$h_{a,b,c}(x) = ax^2 + bx + c$$ 3-wise independent

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- Two points determine a line. Similarly, k points determine a univariate polynomial of degree $k - 1$
- Random degree $k - 1$ polynomials are k-wise independent!
- Practice problem: prove this!

Efficiency

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Efficiency

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- In XOR example, our function takes $O(b)$ storage space, and $O(b)$ time to compute.[a]

  Store set S $O(b \log b)$ get b bits evaluate $\oplus$ Y_i ies

- In $\mathbb{F}_p$ examples, our function takes $O(1)$ storage space and $O(1)$ time to compute![b]

  Store a, b defining $h_{a,b}$ compute $x \mapsto ax+b$

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[a] Reminder that we assume that $b < w$.

[b] We assume that $p < 2^w$.

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  - Coping with randomness
  - Universal Hashing
  - Hashing using 2-universal families
  - Perfect Hashing
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- Computation time of $h_{a,b}$ is also $O(\log m)$
- Can this hash function match chain hashing parameters? ($O(\log \log n)$ search time) ↳ using two random hash functions

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Lemma (Maximum number of collisions)

The expected number of collisions [pairs] *when hashing* ℓ *elements using a 2-universal hash family is*

$$\leq \ell^2/2n$$

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Thus, by Markov's inequality, we have $Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$

Lemma (Maximum load of entry of hash table)

With probability $\geq 1/2$ the ~~number of collisions~~ max load when hashing ℓ elements using a 2-universal hash family is

$$\leq \sqrt{\frac{2\ell^2}{n}}.$$

When $\ell \approx n$ (as is usually assumed in hashing), we expect $\sqrt{2n}$.

Can prove max load is $\sqrt{\text{max \# collisions}}$

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Perfect Hashing

- **Setup:** we are given *in advance* a static set $S \subset U$ of size n
- How to build a hash table with $O(1)$ search time and $O(n)$ memory?
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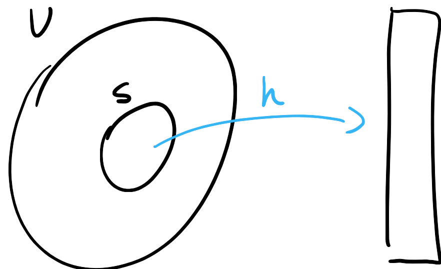
Theorem

The two-level approach gives perfect hashing scheme.

Proof (sketch) of Theorem

- Pick first layer hash function h uniformly at random from our 2-universal family $\mathcal{H}$. Test h on our set S.



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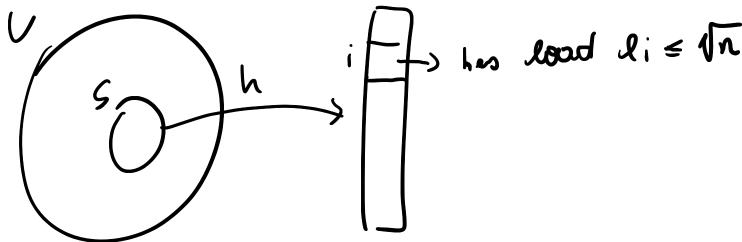

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- By our lemma, if take $h_i : S \to [\ell_i^2]$ from our 2-universal hash family, h_i is perfect with high probability

for S_i

set elements mapped to i

$(|S_i| = \ell_i \leq \sqrt{n})$

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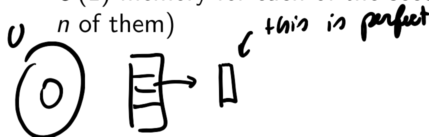

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- *Time to hash*: $O(1)$ time to evaluate each hash function, and we only have two layers. So total time $O(1)$

Acknowledgement

- Lecture based largely on Lap Chi's notes and on [CLRS 2009, Chapter 11].
- See Lap Chi's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L05.pdf

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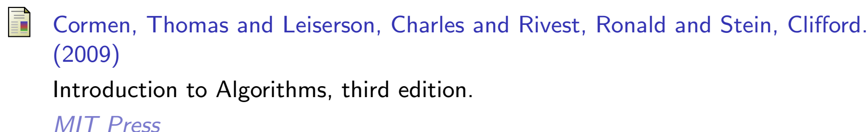
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