Lecture 4: Balls & Bins

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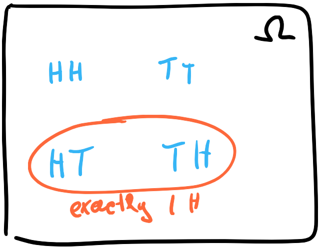
Overview

- Introduction
  - Probability basic notions
  - Balls and Bins
  - Analyses
- Coupon Collector and Power of Two Choices
  - Coupon Collector
  - Power of Two Choices
- Acknowledgements

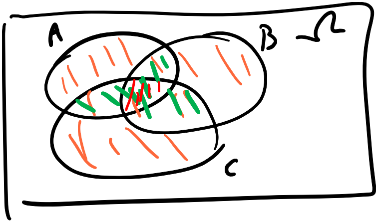
Event Spaces and Inclusion-Exclusion



event space

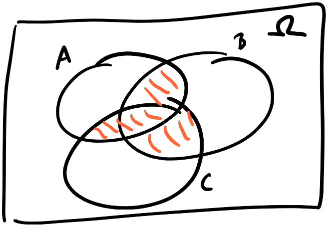
orange region

$$A \cup B \cup C - (A \cap B) - (A \cap C) - (B \cap C) \cup (A \cap B \cap C)$$



$$\Pr[\text{orange event}] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[A \cap C] + \Pr[A \cap B \cap C]$$

Union Bound and Inclusion-Exclusion

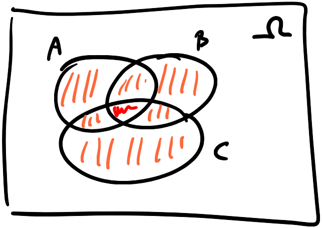


$$\Pr[\geq \text{one of } A, B, C \text{ happen}] \triangleq \Pr[A \cup B \cup C]$$

$$\leq \Pr[A] + \Pr[B] + \Pr[C]$$

counting orange more than once

Union Bound and Inclusion-Exclusion



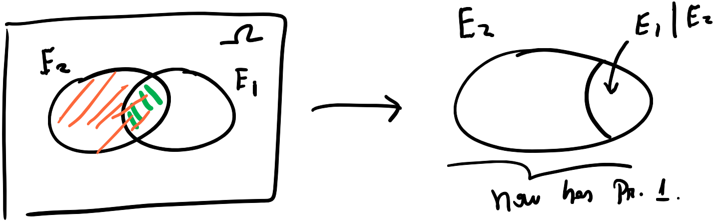
$$\Pr[A \cup B \cup C] \geq \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C]$$

generalize this to any level of intersections.

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is

$$\underbrace{\Pr[E_1 \mid E_2]}_{\text{probability that } E_1 \text{ happens given that } E_2 \text{ happened}} := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$



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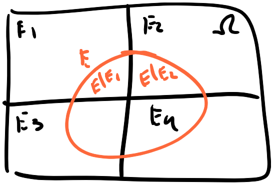
$$\Pr[E] = \sum_{i=1}^{k} \underbrace{\Pr[E \mid E_i] \cdot \Pr[E_i]}_{\Pr[E \cap E_i]}$$



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- **Bayes' rule**: $E_1, \ldots, E_k$ partition our sample space then for event E

$$\Pr[E_i \mid E] = \frac{\Pr[E \cap E_i]}{\Pr[E]} = \frac{\Pr[E \mid E_i] \cdot \Pr[E_i]}{\sum_{j=1}^{k} \Pr[E \mid E_j] \cdot \Pr[E_j]}$$

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Setup: we have *m* balls and we want to put them in *n* bins.

As it is NBA playoff season, we will do this by throwing each ball into a *uniformly random* bin *independently*.

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- What is "typically" the *maximum* number of balls in any bin?
- What is the *expected* number of bins with k balls in them?
- For what values of m do we expect to have *no empty bins*? (coupon collector)

Why Learn About Balls and Bins?

In next lectures, we are going to learn about and analyse *randomized algorithms*. While we will usually analyse the *expected running times* of the algorithms, we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

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Running time *small* with *high probability better than* small expected running time.

In **this lecture**, we will analyse random processes (*balls & bins*) which underlie several randomized algorithms!

Applications ranging from:

1. data structures
2. routing in parallel computers
3. many more!

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$B_{ij} \stackrel{\Delta}{=} \begin{cases} 1 & \text{if } i^{th} \text{ ball falls into bin } j \\ 0, & \text{otherwise} \end{cases}$$

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&= \sum_{i=1}^{m} \Pr\left[\text{ball } i \text{ in bin } j\right]
\end{aligned}$$

$\Pr[B_{ij} = 1] \cdot 1 + 0 \cdot \Pr[B_{ij} = 0]$

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\end{aligned}$$

When $m = n$, expectation of one ball per bin. How often will this actually happen?

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$N_i \triangleq \begin{cases} 1 & \text{if bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$$

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&= \sum_{i=1}^{n} \underbrace{\Pr[\text{bin } i \text{ is empty}]}_{\text{no ball landed in bin } i} = \left(1-\frac{1}{n}\right)^m
\end{aligned}$$

$$\Pr[B_{ki} = 0] = 1 - \frac{1}{n}$$

by independence of ball throwing

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When $m = n$, expected fraction of empty bins is $\frac{1}{e}$.

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

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When $m = n$, second calculation had expectation of $1/e$ *fraction of empty bins*.

Which expectation should I actually "expect"?

As we mentioned earlier, this is where *concentration of probability measure* tries to address. It turns out that the *second random variable* (and thus second calculation) is concentrated around the mean (i.e., expectation).

So we "expect" (or it is "typical") to see around $1/e$-fraction of empty bins when $m = n$

Maximum load in a bin

What is the "typical" maximum number of balls in a bin?

As we saw in the previous slide, "typical" is related to concentration of probability measure.

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As we saw in the previous slide, "typical" is related to concentration of probability measure.

Let us first see a simpler problem, which is known as the *birthday paradox*: for what value of m do we expect to see two balls in one bin?

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \dots \cdot \left(1 - \frac{m-1}{n}\right) \leq e^{-1/n} \cdots e^{-\frac{m-1}{n}} \approx e^{\frac{-m^2}{2n}}$$

1st ball no collision

avoid where 1st ball

avoid 1st & 2nd balls

Birthday Paradox

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This is $\leq 1/2$ when $m = \sqrt{2n \ln(2)}$. For $n = 365$, this is $m \approx 22.4$ for the probability that two people *(balls)* have birthday on the same date *(bins)* to become $\geq 1/2$.

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Thus, expect to see collision (two balls in the same bin) when $m = \Theta(\sqrt{n})$. This appears in several places:

- hashing
- factoring
- many more

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

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$$\Pr[\text{bin 1 has } \geq k \text{ balls}] \leq \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}]$$

because $m = n$

Union bound

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$$= \sum_{\substack{S \;\; subset[n] \\ |S|=k}} \prod_{i \in S} \frac{1}{n}$$

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&= \binom{n}{k} \cdot \frac{1}{n^k} \leq \left(\frac{ne}{k}\right)^k \cdot \frac{1}{n^k} = \frac{e^k}{k^k}
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By union bound

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq \sum_{i=1}^{n} \Pr[\text{bin i has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k}$$

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

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$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

complement

||

all bins have $\leq k$ balls

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When $k \ln k > \ln n$. Setting $k = 3\dfrac{\ln n}{\ln \ln n}$ does it.

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This comes up in hashing and in analysis of approximation algorithms (for instance, best known approximation ratio for congestion minimization).

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Why is this problem called the coupon collector problem?

Because we can formulate it in the following way:

- suppose each bin is a different coupon
- we buy one coupon at random (like kinder eggs/pack action cards)
- what is the number of coupons that we need to buy to collect all of them?

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Let X_i be the number of balls thrown to get from $i+1$ empty bins to i empty bins. Let X be the number of balls thrown until we have no empty bins.

$$X = \sum_{i=0}^{n-1} X_i$$

Coupon Collector

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$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right]$$

linearity of expectation

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What is $\mathbb{E}[X_i]$?

X_i geometric random variable with parameter $\boxed{p = \frac{i}{n}.}$

Number of trials until the first success, where success probability p.

when we throw ball into empty bin
i empty bins

$$\Pr[X_i = k] = \underbrace{(1-p)^{k-1}}_{\text{Pr we failed in attempts } 1,\dots,k-1} \cdot \underbrace{p}_{\text{succeed at } k}$$

Coupon Collector - Computing $\mathbb{E}[X]$

X_i takes values over $\mathbb{N}^*$

$$\mathbb{E}[X_i] = \sum_{k=1}^{\infty} k \cdot \Pr[X_i = k] \underset{\substack{\uparrow \\ \text{rearrange}}}{=} \sum_{k=1}^{\infty} \underbrace{\Pr[X_i \geq k]}_{(1-p)^{k-1}} =$$

$X_i \geq k-1$

$X_i \geq k-2$

$\vdots$

$X_i \geq 1$

$$= \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p} = \frac{n}{i}$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{i} = n \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{H_n} \approx n \log n$$

Coupon Collector - Computing $\mathbb{E}[X]$

This $n \ln n$ bound shows up in:

- cover time of random walks in complete graph
- number of edges needed in graph sparsification
- many more places

Power of Two Choices

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Intuition/idea: let the height of a bin be the # balls in that bin. This process tells us that to get one bin with height $h + 1$ we must have at least two bins of height h.

We can bound # bins with height at least h (because this will tell us how likely it is to get to height $h + 1$).

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A bit more intuition

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How do we turn this into a proof?

Proof Sketch (proof [Mitzenmacher & Upfal, Chapter 14])

Use following Chernoff bound on binomial random variable $B(n, p)$ with n trials and success probability p.[2]

$$\Pr[B(n,p) \geq 2np] \leq e^{-np/3}$$

[2] That is, $\Pr[B(n,p) = k] = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$.

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- $\Pr[E(4, n)] = 1$ (why?)

$E(4,n)$ = event that after all n balls thrown $N_4 \leq \frac{n}{4}$

$N_4 :=$ # bins with $\underbrace{\text{4 balls}}_{\text{height} \geq 4}$

n = # balls $= 1 \cdot N_1 + 2 \cdot N_2 + 3 \cdot N_3 + 4 \cdot N_4 + \cdots \geq 4 \cdot N_4$

$\Rightarrow N_4 \leq n/4$

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- $E(h, t) :=$ event that after all t balls are thrown, $N_h \leq \beta_h$
- $\Pr[E(4, n)] = 1$
- We will prove that if $E(h, n)$ holds with high probability then so does $E(h+1, n)$ (so long as h is "small enough")

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Proof Sketch (proof [Mitzenmacher & Upfal, Chapter 14])

- $Y_t(h)$ be the indicator variable that t^{th} ball has height $\geq h+1$ (i.e., was placed in a bin that had height h)
- $\Pr[Y_t(h) = 1 \mid E(h,t)] \leq \left(\frac{N_h}{n}\right)^2 \leq \frac{\beta_h^2}{n^2}$

  if this happens then $N_h \leq \beta_h$
- If $p_h := \frac{\beta_h^2}{n^2}$ then

$$\Pr\left[\sum_{t=1}^{n} Y_t(h) > k \mid E(h,n)\right] \leq \Pr\left[B(n,p_h) > k \mid E(h,n)\right]$$

-

  \# bins of height $\geq h+1$ \# balls have height $h+1$

$$\Pr[N_{h+1} > k \mid E(h,n)] \leq \Pr\left[\sum_{t=1}^{n} Y_t(h) > k \mid E(h,n)\right]$$
$$\leq \Pr\left[B(n,p_h) > k \mid E(h,n)\right]$$

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$$\begin{aligned}
\Pr[N_{h+1} > \beta_{h+1} \mid E(h,n)] &\leq \Pr\left[B(n,p_h) > \beta_{h+1} \mid E(h,n)\right] \\
&\leq \frac{\Pr\left[B(n,p_h) > \beta_{h+1}\right]}{\Pr[E(h,n)]} \\
&\leq \frac{1}{\Pr[E(h,n)] \cdot e^{np_h/3}} \qquad \text{(Chernoff)}
\end{aligned}$$

bad event

Proof Sketch (proof [Mitzenmacher & Upfal, Chapter 14])

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Setting $k = \beta_{h+1} = 2np_h$ above, we get

$$\begin{aligned}
\Pr[\underbrace{N_{h+1} > \beta_{h+1}}_{\text{bad event}} \mid E(h,n)] &\leq \Pr\left[B(n,p_h) > \beta_{h+1} \mid E(h,n)\right] \\
&\leq \frac{\Pr\left[B(n,p_h) > \beta_{h+1}\right]}{\Pr[E(h,n)]} \\
&\leq \frac{1}{\Pr[E(h,n)] \cdot e^{np_h/3}} && \text{(Chernoff)}
\end{aligned}$$

Thus, setting $p_h \cdot n \geq 6 \ln n$ we get

$$\Pr[\underline{\text{not } E(h+1,n)} \mid E(h,n)] = \Pr[N_{h+1} > \beta_{h+1} \mid E(h,n)] \leq \frac{1}{n^2 \Pr[E(h,n)]}$$

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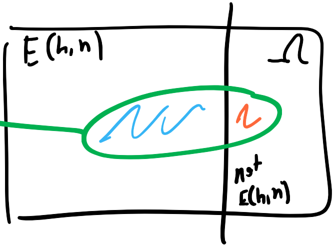
$$\Pr[\text{not } E(h+1,n)] = \Pr[\text{not } E(h+1,n) \mid E(h,n)] \cdot \Pr[E(h,n)] + \Pr[\text{not } E(h+1,n) \mid E(h,n)] \cdot \Pr[\text{not } E(h,n)]$$



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&\quad + \Pr[\underbrace{\text{not } E(h+1,n)}_{\le 1} \mid E(h,n)] \cdot \Pr[\underbrace{\text{not } E(h,n)}_{\text{small}}] \\
&\le \frac{1}{n^2} + \Pr[\text{not } E(h,n)] \qquad (\text{so long as } p_h n \ge 6 \ln n)
\end{aligned}$$

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\end{aligned}$$

To finish the proof, need to show:

- $p_h \cdot n \geq 6 \ln n$ for $h = O(\ln \ln n)$ (easy calculation - we did it)
- Handle the case where $p_h \cdot n < 6 \ln n$. (another Chernoff bound - see Lap Chi's notes)

Acknowledgement

- Lecture based largely on Lap Chi's notes and on [Motwani & Raghavan 2007, Chapter 3].
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L04.pdf`

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