Lecture 1: Amortized Analysis & Union Find

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Overview

- Introduction
  - Why amortized analysis?
  - Types of amortized analyses
  - Union-Find
- Implementing Union-Find
  - Setup
  - First approach
  - Tree Representation & Path Compression
  - Analysis
- Acknowledgements

Why Amortized Analysis?

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Worst or average-case complexity of data structures

Data Structure	search	insertion	deletion
Doubly-Linked List	$O(n)$	$O(1)$	$O(n)$
Ordered Array	$O(\log n)$	$O(n)$	$O(n)$
Hash Tables[a]	$O(1)$	$O(1)$	$O(1)$
Balanced Binary Search Trees[b]	$O(\log n)$	$O(\log n)$	$O(\log n)$

[a] Average-case, although worst-case search time is $\Theta(n)$

[b] Also average-case. Worst-case complexity is $O(height)$ of the tree, which can be $\Theta(n)$.

Why Amortized Analysis?

In **amortized analysis**, one averages the *total time* required to perform a sequence of data-structure operations over *all operations performed*.

Upshot of amortized analysis: worst-case cost *per query* may be high for one particular query, so long as overall average cost per query is small in the end!

Remark

Amortized analysis is a *worst-case* analysis. That is, it measures the average performance of each operation in the worst case.

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2. **Accounting Method:** assign certain *charge* to each operation (independent of the actual cost of the operation). If operation is cheaper than the charge, then build up credit to use later.
3. **Potential Method:** one comes up with *potential energy* of a data structure, which maps each state of entire data-structure to a real number (its "potential"). Differs from accounting method because we assign credit to the data structure as a whole, instead of assigning credit to each operation.

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1. Find the unique set containing a particular element
   1. Input: element v from universe of elements
   2. Output: set containing v
2. Take union of two disjoint sets
   1. Input: two sets A, B from current collection of sets
   2. Output: updated collection of sets, i.e. with $A \cup B$ and without A, B

Application: Kruskal's minimum spanning tree algorithm

Input: graph $G(V, E)$ and edge weights $w : E \to \mathbb{N}$
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    1. if endpoints of e_i in different connected components of T (use two *find operations* on endpoints of e_i to check this step)
        - $T \leftarrow T \cup \{e_i\}$
        - combine the connected components of endpoints of e_i (*union operation*)

$e_i = \{u, v\}$

$\text{FIND}(u) \overset{?}{=} \text{FIND}(v)$

same connected component

$\neq$

disjoint connected components

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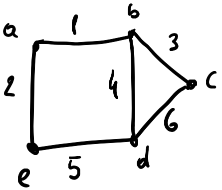
Remark

In this application, we care about the *total cost* of all operations (unions and finds). Thus, amortized analysis is better than worst-case *per query*.

Example



$T = \emptyset$ $\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$ $\{e\}$

1. FIND(a) $\neq$ FIND(b)

$T = \{\{a,b\}\}$ $\underbrace{\{a,b\} \quad \{c\} \quad \{d\} \quad \{e\}}_{\text{UNION}(S_a, S_b)}$

2. FIND(a) $\neq$ FIND(e)

$T = \{\{a,b\}, \{a,e\}\}$

$\{a,b,e\}$ $\{c\}$, $\{d\}$

$T = \{\{a,b\}, \{a,e\}, \{b,c\}\}$

$\{a,b,c,e\}$ $\{d\}$

. . .

Example (continued)

Setup

Notation:

- $n \leftarrow$ number of elements (we denote the elements by $1, 2, \ldots, n$)
- $m \leftarrow$ number of operations. That is

$$m = (\text{number of finds}) + (\text{number of unions})^1$$

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- $FIND(k) \leftarrow$ find the set containing element k
- $UNION(A, B) \leftarrow$ updates data structure by deleting sets A, B and constructing $A \cup B$

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Naive approach

Keep an array S of size n where

$$S[i] \text{ contains the name of set containing element } i.$$

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No amortized analysis yet.

What if when taking the union of A and B, we only change name of the set of least size?

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Cost of all unions $= O(n \log n)$, as for each element $i \in \{1, \ldots, n\}$, we have that the *UNION* operation will change $S[i]$ at most $\log n$ times.

Proof.

Every time we change $S[i]$, the size of the set containing element i doubles. $\square$

$\{i\} \longrightarrow$

1 2 4 . -

$2^{\log n} = n$

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Proof.

Every time we change $S[i]$, the size of the set containing element i at least doubles. □

Thus, cost of m operations is $O(m + n \log n)$ and we get that amortized cost is $O\left(1 + \frac{n \log n}{m}\right)$. If $m = \Omega(n \log n)$ this is best possible.

$m = \# \text{FINDS} + \# \text{UNIONS}$

$$\frac{T(m)}{m} = \frac{(\# \text{FINDS}) \cdot 1 + n \log n}{m}$$

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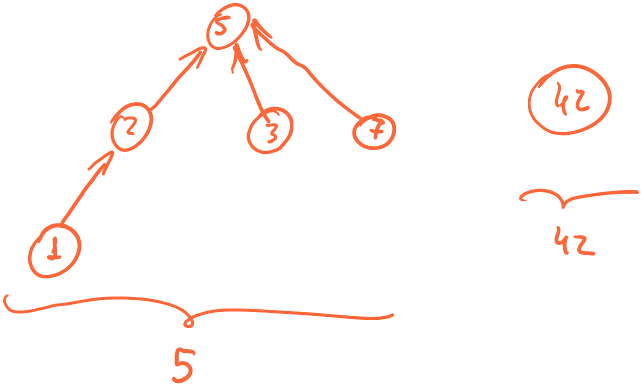
Are we done? What if $m = o(n \log n)$, can we do better?

Tree Representation

Represent each set as a tree of parent pointers. Each set will have its root as its representative element.



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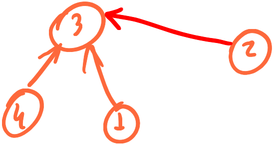
Union(2,3)



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Question

How to define "smaller" (i.e., the "size" of a tree)?

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Question

How to define "smaller" (i.e., the "size" of a tree)?

- What if we define the size of a tree to be number of elements?
- What if we define the size of a tree to be it's height (longest path from leaf to root)?

Bad instances

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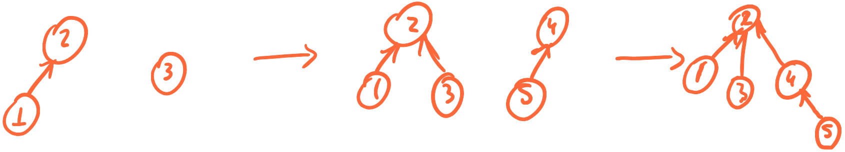
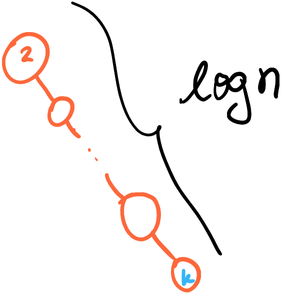




total time of FIND $O(m \log n)$

Bad instances

- What if we define the size of a tree to be it's height (longest path from leaf to root)?

similar to previous slide

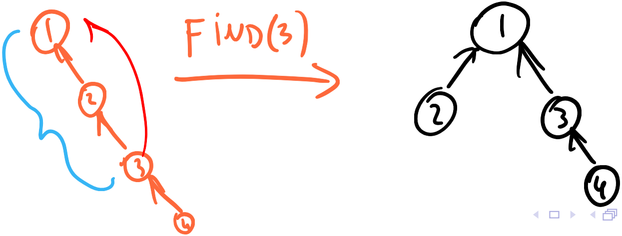
Path Compression

- Problems above arise because our trees are *static*
- leads to bad FIND operations
- need to make all trees "flat." (or at least the *most queried items*)

Definition (Path compression)

After each $FIND(k)$, for every node j on path $k \to \cdots \to \text{root}$, set

$$PARENT(j) \leftarrow \text{root}.$$



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This messes up the height of the tree, as path compression may change it.

Rank of a tree

Definition (Rank of tree)

For each tree with root r, define rank(r) as follows:

- if the tree is a single element (r in this case) rank$(r) = 0$
- when performing union of two trees with roots r_1, r_2, if rank$(r_1) \geq$ rank(r_2), then
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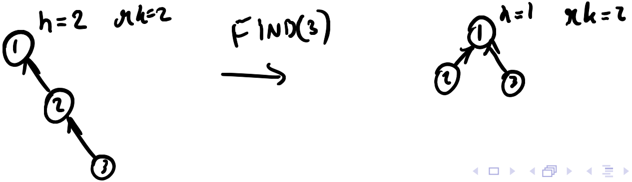
Intuition: rank of a tree is the *height* if *no path compressions* had been done.



Final Algorithm

Input: set of elements $\{1, 2, \ldots, n\}$
Output: at each step, a union-find data structure comprised of disjoint union of sets whose union is $\{1, 2, \ldots, n\}$

1. Start with each set being $\{k\}$, where $k \in \{1, \ldots, n\}$. Set $\text{rank}(k) = 0$.
2. $UNION(S_1, S_2)$: where r_1, r_2 are the roots of S_1, S_2
   if $\text{rank}(r_1) \geq \text{rank}(r_2)$:
   1. make $\text{root}(S_1 \cup S_2) = r_1$, by creating pointer $r_2 \to r_1$.
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Analysis

Theorem ([Tarjan 1975])

The amortized cost per operation of union-find is $\Theta(\alpha(m, n))$*, where* $\alpha(m, n)$ *is the inverse Ackermann function. That is, the (worst-case) cost of* m *operations is* $\Theta(m \cdot \alpha(m, n))$*.*

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Remark

Inverse Ackermann function is mega-hyper-super slow growing. For more about the Ackermann function and its inverse, see `https://en.wikipedia.org/wiki/Ackermann_function`.

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$$\log^*(n) := \min\{i \mid \log^{(i)}(n) \leq 1\},$$

where $\log^{(i)}$ means that we apply the log function i times.

n	1	2	$3, 4 = 2^2$	$5, \ldots, 16 = 2^{2^2}$	$17, \ldots, 65536 = 2^{16}$
$\log^*(n)$	0	1	2	3	4

$\log^{(2)} n = \log \log n$

$\log^{(3)} n = \log \log \log n$

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In the accounting method, we need to choose a charge to each operation $\hat{c}_i$ such that

total charge

$$\sum_{i=1}^{\ell} \hat{c}_i \geq \sum_{i=1}^{\ell} c_i$$

total cost of operation

for all $\ell \leq m$, where c_i is the actual cost of the i^{th} operation.

Final Algorithm - recap

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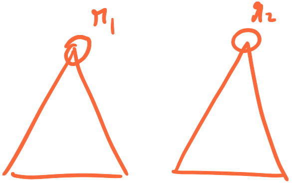
Claim

When an element k is assigned $\mathrm{rank}(k) = r$ *then k has* $\geq 2^r$ *descendants.*

Proof sketch: induction

base case $r = 0 \Rightarrow$ singleton $\therefore$ ✓



$r_1' = r_1 + 1$

$$2^{r_1} + 2^{r_2} \leq 2^{r_1+1} = 2^{r_1'}$$

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$\text{rank}(k) < \text{rank}(\textit{parent}(k))$

follows from definition of union

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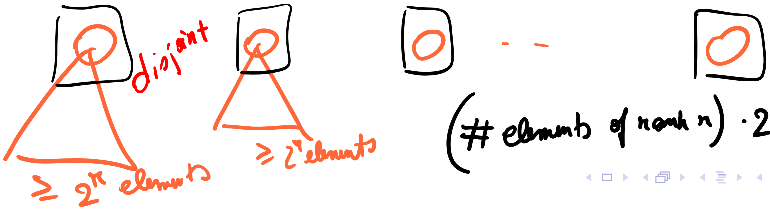
Claim

Number of vertices of rank r is $\leq n/2^r$.



$(\#\text{ elements of rank } r) \cdot 2^r \leq n$

Grouping Elements Based on Rank

Idea: divide vertices into groups based on rank.

Element of rank r goes into group $\log^*(r)$. In particular, for element k, we have:

$$group(k) := \log^*(\mathrm{rank}(k))$$

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Remark

Number of groups: $\log^*(n)$.

Analysis - Charging Scheme Idea

Actual cost of $FIND(k)$: distance from k to root.
Idea: charge some of this cost to *FIND* and some to nodes along path.

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Idea: charge some of this cost to $FIND$ and some to nodes along path.

Charging scheme:

1. $FIND(k)$
   - For each element u in the path $k \to$ root:
     - if u has parent and grandparent in path and group(u) = group(parent(u)), then charge 1 to u
     - else charge 1 to $FIND(k)$.
2. $UNION(A, B)$: just charge 1 to this operation

Analysis - Charging Scheme Idea

Actual cost of $FIND(k)$: distance from k to root.

Idea: charge some of this cost to $FIND$ and some to nodes along path.

Charging scheme:

1. $FIND(k)$
   - For each element u in the path $k \rightarrow$ root:
     - if u has parent and grandparent in path and group(u) = group(parent(u)), then charge 1 to u
     - else charge 1 to $FIND(k)$.
2. $UNION(A, B)$: just charge 1 to this operation

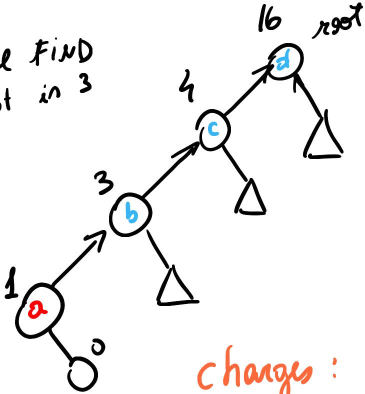
Remark

Note that charging scheme for $FIND(k)$ and nodes covers the actual cost of $FIND(k)$, since we are charging either the node on the path or the operation $FIND(k)$.

Since charging for $UNION$ also covers the cost of the union operation, we have a valid charging scheme.

Example of Charging Scheme for FIND

total FIND cost in 3



FIND(a) $\longrightarrow$

$\text{group}(a) = 0$
$\text{group}(b) = 2$
$\text{group}(c) = 2$
$\text{group}(16) = 3$

changes :

$\tilde{c}(\text{FIND}) = \overset{a}{1} + \overset{b}{0} + 1$ $\qquad \tilde{c}(d) = 0$

$\tilde{c}(a) = 0$
$\tilde{c}(b) = 1$
$\tilde{c}(c) = 0$

Example of Charging Scheme for FIND

Charging Scheme Formally

(similar to first algorithm)

at each step

total bound over # ops.

So, how do we define the charges to $FIND(k)$?

$$\underbrace{\hat{c}_i(FIND(k))}_{\text{charge from charging scheme}} = \underbrace{\boxed{\tilde{c}_i(FIND(k))} + \sum_{u \in \text{ path } k \to u} \boxed{(\text{charge to } u)}}_{\text{broken down charges to analyze algorithm}}$$

Analysis

Now we need to analyse the total amortized cost of this charging scheme.

Analysis

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- Total charge to each $FIND(k)$ is $\leq \log^*(n) + 1$
  - Group changes $\leq \log^*(n) - 1$ times
  - $+2$ for root of tree and child of root of tree

$\boxed{\tilde{c}_i(FIND)}$

to change FIND, we need to
either change group (group needs to increase)

or
we don't have parent and grandparent

Analysis

Now we need to analyse the total amortized cost of this charging scheme.

- Total charge to each $FIND(k)$ is $\leq \log^*(n) + 1$
  - Group changes $\leq \log^*(n) - 1$ times
  - $+2$ for root of tree and child of root of tree
- Total charge to each element $x \in \{1, \ldots, n\}$:

  rank increase
  - if x is charged in a path compression, then x is not root and path compression will give it a parent of higher rank than old parent.
  - if x has a parent in a higher group, then x will no longer be charged.
  - thus, if group$(x) = g$ then x can be charged at most

$$(\text{number of ranks in group } g) - 1 \leq 2 \uparrow g$$

$2 \uparrow 1 = 2 \qquad 2 \uparrow 2 = 2^2 \qquad 2 \uparrow 3 = 2^{2^2}$

Analysis

Now we need to analyse the total amortized cost of this charging scheme.

- Total charge to each $FIND(k)$ is $\le \log^*(n) + 1$
  - Group changes $\le \log^*(n) - 1$ times
  - $+2$ for root of tree and child of root of tree
- Total charge to each element $x \in \{1, \dots, n\}$:
  - if x is charged in a path compression, then x is not root and path compression will give it a parent of higher rank than old parent.
  - if x has a parent in a higher group, then x will no longer be charged.
  - thus, if group$(x) = g$ then x can be charged at most

$$(\text{number of ranks in group } g) - 1 \le 2 \uparrow g$$

Let $N(g)$ be number of elements in group g. Then

$$N(g) \le \underbrace{\sum_{r=2\uparrow(g-1)+1}^{2\uparrow g} \boxed{\frac{n}{2^r}}}_{\text{sum over \# elements of rank } r \text{ s.t. } \log^*(r)=g.} \le \frac{n}{2^{2\uparrow(g-1)+1}} \cdot \sum_{0}^{\infty} 1/2^i = \frac{n}{2 \uparrow g}$$

Analysis

1. Thus, total charge to all elements in group g:

max

$$(\text{total charge per element in group } g) \cdot N(g) \leq \underline{(2 \uparrow g)} \cdot \frac{n}{2 \uparrow g} = n$$

super small !

Analysis

1. Thus, total charge to all elements in group g:

$$(\text{total charge per element in group } g) \cdot N(g) \leq (2 \uparrow g) \cdot \frac{n}{2 \uparrow g} = n$$

2. Total charge to all elements of $\{1, \ldots, n\}$:

$$(\text{charge to all elements in group } g) \cdot \underbrace{(\text{number of groups})}_{\log^* n} \leq \boxed{n \cdot \log^*(n)}$$

Analysis

1. Thus, total charge to all elements in group g:

$$(\text{total charge per element in group } g) \cdot N(g) \leq (2 \uparrow g) \cdot \frac{n}{2 \uparrow g} = n$$

2. Total charge to all elements of $\{1, \ldots, n\}$:

$$(\text{charge to all elements in group } g) \cdot (\text{number of groups}) \leq n \cdot \log^*(n)$$

3. Total charge to all *FIND* operations:

$$\underbrace{(\text{number of } \mathit{FIND} \text{ operations})}_{\leq m} \cdot \underbrace{(\text{charge per } \mathit{FIND})}_{\leq \log^* n + 1} \leq m \cdot (\log^*(n) + 1)$$

Analysis

1. Thus, total charge to all elements in group g:

$$(\text{total charge per element in group } g) \cdot N(g) \leq (2 \uparrow g) \cdot \frac{n}{2 \uparrow g} = n$$

2. Total charge to all elements of $\{1, \ldots, n\}$:

$$(\text{charge to all elements in group } g) \cdot (\text{number of groups}) \leq n \cdot \log^*(n)$$

3. Total charge to all *FIND* operations:

$$(\text{number of } \mathit{FIND} \text{ operations}) \cdot (\text{charge per } \mathit{FIND}) \leq m \cdot (\log^*(n) + 1)$$

4. Total charge overall: sum of 2 + 3.

$$O((m+n)\log^* n) = O(m \log^* n), \text{ as we assumed } n \leq m$$

Acknowledgement

Lecture based largely on Anna Lubiw's notes. See her notes at https://www.student.cs.uwaterloo.ca/~cs466/Lectures/Lecture5.pdf

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