## Problem 1

Given a 3-uniform hypergraph $G(V, E)$ (that is, a hypergraph where each hyperedge has exactly 3 vertices), we say that a 2-coloring of $V$ is valid for a hyperedge $e = \{a, b, c\} \in E$ if the hyperedge $e$ is not monochromatic upon this coloring.

The Max-2C3U problem is the following:

- **Input:** a 3-uniform hypergraph $G(V, E)$
- **Output:** a 2-coloring of the vertices of $G$ of maximum value, that is, a function $f : V \to \{-1, 1\}$ (the coloring) which maximizes the number of valid hyperedges.

In this question, you are asked to:

1. Write the optimization problem above as a quadratic program
2. Formulate an SDP relaxation for the problem, and prove that it is in fact a relaxation

## Problem 2

Given a graph $G$, an *independent set* is a set of vertices such that no two are neighbors. The Maximum Independent Set (MIS) problem is a famous NP-complete problem.

Interestingly, the complement of an independent set is a vertex cover, so the complement of the MIS is the minimum vertex cover. We've seen (twice) how to get a two-approximation for vertex cover. Even though it is complementary, the situation for MIS is much worse.

Early in the study of approximation hardness, MIS was shown to be MAX-SNP-hard, meaning there is some constant to within which it *cannot* be approximated (unless P = NP).

Suppose one has an $\alpha$-approximation algorithm for MIS. Consider the following "graph product" operation for a graph $G$. Create a distinct copy $G_v$ of $G$ for each vertex $v$ of $G$. Then connect up the copies as follows: if $(u, v)$ is an edge of $G$, then connect every vertex in $G_u$ to every vertex in $G_v$.

(a) Prove that if there is an independent set of size $k$ in $G$, then there is an independent set of size $k^2$ in the product graph.

(b) Prove that given an independent set of size $s$ in the product graph, one can find an independent set of size $\sqrt{s}$ in $G$.

(c) Conclude from the MAX-SNP-hardness of MIS that MIS has *no* constant-factor approximation (unless P = NP).

## Problem 3

Prove the following relations directly from the definitions of **PCP**, **IP**, **NP**, **co-RP** and **P**:

1. $NP \subseteq IP$
2. co-$RP \subseteq IP$
3. $P = PCP[0, 0]$
4. $NP = PCP[0, \mathsf{poly}(n)]$
5. co-$RP = PCP[\mathsf{poly}(n), 0]$

## Problem 4

You have reached a river (modelled as a straight line) and must find a bridge to cross it. The bridge is at some integer coordinate upstream or downstream.

1. Give a 9-competitive deterministic algorithm for optimizing the total distance travelled up and downstream before you find the bridge.[1]
2. Give a randomized 7-competitive algorithm for the problem

## Problem 5

A *conservative algorithm* is one that makes at most $k$ page faults on any consecutive subsequence of the input that contains at most $k$ pages. Here $k$ is the size of the cache.

1. Prove that **LRU** and **FIFO** are conservative
2. Prove that any conservative algorithm is k-competitive.

## Problem 6

Given a matrix $A \in \mathbb{Q}^{N\times N}$ where $N = 2^k$, prove that one can compute $\det(A)$ in time $O(N^{\omega})$, where $\omega$ is the matrix multiplication exponent. You can assume that any matrix that you need to invert in the process is invertible.

**Optional question:** how would you remove the assumption given above?

[1] This is optimal for deterministic strategies.