Lecture 22: Zero-Knowledge Proofs

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Overview

- Administrivia
- Why Zero Knowledge?
- Zero-Knowledge Proofs
- Conclusion
- Acknowledgements

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from *November 24th until December 7th* and provide us with your evaluation and feedback on the course!

- This would really help me figuring out what worked and what didn't for the course

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- Undergraduate Research Fellowship (URF):

  `https://grec.cs.uwaterloo.ca/`

- Undergraduate Research Internship (URI):

  `https://cs.uwaterloo.ca/current-undergraduate-students/research-opportunities/undergraduate-research-internship-uri-program`

- For Canadians, please check out NSERC's USRA:

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  - But then Bob has access to her entire database!
  - Can Alice convince Bob that she gave right file without giving any more *knowledge* beyond that she gave right file?

Zero-Knowledge Proofs

Proofs in which the verifier gains *no knowledge* beyond the validity of the assertion.

How do you prove that you know something without showing how you know it.

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- In both cases Alice conveyed *information*!

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- Verifier *does not trust* prover. Otherwise no need to verify proof!

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C = graph G is Hamiltonian

P := give a Hamiltonian cycle

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Example: Graph Isomorphism

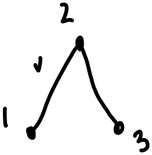
- Setup:
  - A claim $\mathcal{C} :=$ graphs G_0, G_1 are isomorphic

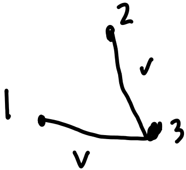
iff there is a permutation of vertices

$$\sigma : V_{G_0} \to V_{G_1}$$

s.t.

$$\{u, v\} \in E_{G_0} \iff \{\sigma(u), \sigma(v)\} \in E_{G_1}$$





$\sigma(1) = 1$

$\sigma(2) = 3$

$\sigma(3) = 2$

isomorphic

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- In this setting, verifier *learns the isomorphism* (i.e., the proof)!

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  - Verifier is allowed *private randomness*
- In the end, we will see a (zero-knowledge) proof for graph isomorphism as follows:

  Alice: I will not give you an isomorphism, but I will prove that I could give you one, if I wanted to.

G_0 G_1 isomorphic ($\underline{\rho}$ isomorphism)

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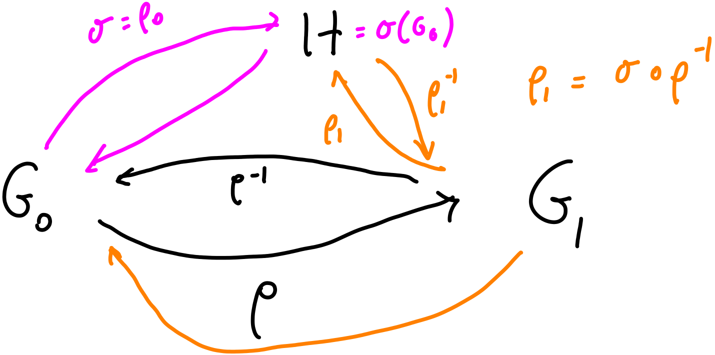

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$$\rho_b = \rho \circ \tilde{\pi}$$

random permutation!

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$G_0 \leftrightarrow H$

$b = 1$ then prover's done

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- How can we model the fact that verifier does not gain knowledge?!

Simulation!

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- Simulation $\Rightarrow$ V gained no new information!

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Definition (Perfect Zero Knowledge)

A prover P is *perfect zero-knowledge* for language L if for every polynomial time, randomized verifier V^*, there is a randomized algorithm M^* such that for every $x \in L$ the following random variables are identically distributed:

- $\langle P, V^* \rangle(x)$, that is, output of interaction between prover P and verifier V^* on input x
- $M^*(x)$, that is, output of algorithm M^* (simulation) on input x

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Note that we usually talked about not trusting provers so far, but for Zero-Knowledge, we will *not trust verifiers* (as they may try to obtain information about the proof!)

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A prover P is *perfect zero-knowledge* for language L if for every polynomial time, randomized verifier V^*, there is a randomized algorithm M^* such that for every $x \in L$ the following random variables are identically distributed:

- $\langle P, V^* \rangle(x)$, that is, output of interaction between prover P and verifier V^* on input x
- $M^*(x)$, that is, output of algorithm M^* (simulation) on input x

- The above captures the idea that V^* is not gaining any extra computational power by interacting with P, since same output could have been generated by M^*

Perfect Zero Knowledge Proof[2]

- Previous definition is a bit too strict to be useful, so we relax it.[1]
- We will allow simulator to fail with small probability (denoted by outputting $\perp$)

[1] Very common phenomenon in crypto, that statistical indistinguishability too strict.
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1. With probability $\leq 1/2$, $M^*(x) = \perp$ fails
2. Conditioned on $M^*(x) \neq \perp$, the following variables are identially distributed: not failing
   - $\langle P, V^* \rangle (x)$, that is, output of interaction between prover P and verifier V^* on input x
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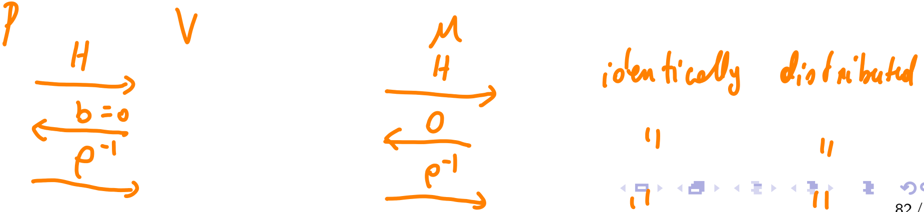

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- Note that whenever we don't fail, we output same distribution as the original protocol!

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- We saw today how the power of interaction can be used to verify validity of “proofs” without conveying information about it
- Has applications in
  - Modern cryptography
  - Credit Cards
  - Passwords
  - Complexity Theory (can use zero-knowledge to construct complexity classes)
  - Used in cryptocurrencies (validate transactions without giving details about transactions)

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  - Oded Goldreich's Foundations of Cryptography book, Chapter 6
  - Berkeley & MIT's 6.875 Lecture 14

    https://inst.eecs.berkeley.edu/~cs276/fa20/slides/lec14.pdf