Lecture 21: Matrix Multiplication & Exponent of Linear Algebra

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Overview

- Administrivia
- Matrix Multiplication
- The Exponent of Linear Algebra
- Matrix Inversion
- Determinant and Matrix Inverse
- Conclusion
- Computing Partial Derivatives

Rate this course!

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- This would really help me figuring out what worked and what didn't for the course
- And let the school know if I was a good boy this term!
- Teaching this course is also a learning experience for me :)

How can I learn more?

Consider taking more advanced courses next term!
See graduate course openings at:

- Current graduate course offerings for next term!

  `https://cs.uwaterloo.ca/current-graduate-students/courses/current-course-offerings/winter-2022-tentative`

- Or, try out some of the research opportunities at UW!

URA , URF , USRA

Matrix Multiplication

- **Input:** matrices $A, B \in \mathbb{F}^{n \times n}$
- **Output:** product $C = AB$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21}$$

Matrix Multiplication

- **Input:** matrices $A, B \in \mathbb{F}^{n \times n}$
- **Output:** product $C = AB$
- Naive algorithm:

Compute n matrix vector multiplications.

$$A\begin{pmatrix} | & | & & | \\ B_1 & B_2 & \cdots & B_n \\ | & | & & | \end{pmatrix}$$

Matrix Multiplication

"word-ram model"

- **Input:** matrices $A, B \in \mathbb{F}^{n \times n}$
- **Output:** product $C = AB$
- Naive algorithm:

  Compute n matrix vector multiplications.

- Running time: $O(n^3)$

  Can we do better?

What is the limit of better?

has to be $\Omega(n^2)$ (because have to read input $2n^2$ size)

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Can we do better?

- Strassen 1969: YES!
- Idea: divide matrix into blocks, and *reduce number of multiplications* needed!

Addition is very cheap

Multiplication is expensive ← minimize # of multiplications

Strassen's Algorithm

- Suppose that $n = 2^k$
- Let $A, B, C \in \mathbb{F}^{n \times n}$ such that $C = AB$. Divide them into blocks of size $n/2$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$
$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$
$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$
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$$M(n) \leq 8 \cdot M(n/2) + c \cdot n^2$$

$$M(n) = O(n^3)$$

can we use < 8 multiplications?

Strassen's idea: inspired by Karatsuba's algorithm for polynomial multiplication

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- Define following matrices:

$$S_1 = A_{21} + A_{22},\ S_2 = S_1 - A_{11},\ S_3 = A_{11} - A_{21},\ S_4 = A_{12} - S_2$$

$$T_1 = B_{12} - B_{11},\ T_2 = B_{22} - T_1,\ T_3 = B_{22} - B_{12},\ T_4 = T_2 - B_{21}$$

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- Define following matrices:

$$\frac{n}{2}\times\frac{n}{2} \left\{ \begin{array}{c} S_1 = A_{21} + A_{22},\ S_2 = S_1 - A_{11},\ S_3 = A_{11} - A_{21},\ S_4 = A_{12} - S_2 \\ T_1 = B_{12} - B_{11},\ T_2 = B_{22} - T_1,\ T_3 = B_{22} - B_{12},\ T_4 = T_2 - B_{21} \end{array} \right.$$

- Compute the following 7 products:

$$\frac{n}{2}\times\frac{n}{2} \left\{ \begin{array}{c} P_1 = A_{11}B_{11},\ P_2 = A_{12}B_{21},\ P_3 = S_4 B_{22},\ P_4 = A_{22}T_4 \\ P_5 = S_1 T_1,\ P_6 = S_2 T_2,\ P_7 = S_3 T_3 \end{array} \right.$$

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- $C_{11} = \underline{A_{11}B_{11}} + \underline{A_{12}B_{21}} = P_1 + P_2$

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$$\cancel{A_{11}B_{11}} + (A_{12} + \cancel{\underline{A_{11}}} - \cancel{S_1})B_{22} + \cancel{S_1}(\underline{B_{12} - B_{11}})$$

$$+ (\cancel{S_1} - \underline{A_{11}})(\cancel{B_{22}} - \underline{B_{12}} + \underline{B_{11}}) = C_{12}$$

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- Correctness follows from the computations

Analysis of Strassen's Algorithm

- To compute $AB = C$ we used:
  1. 8 additions → S_i, T_i's
  2. 7 multiplications P_i's
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- Recurrence:

$$MM(n) \le \underbrace{7 \cdot MM(n/2)}_{\text{7 products}} + \underbrace{18 \cdot c \cdot (n/2)^2}_{\text{time it takes to add } \frac{n}{2} \times \frac{n}{2} \text{ matrices}}$$

$$\boxed{n = 2^k}$$

$$MM(2^k) \le 7 \cdot \underline{MM(2^{k-1})} + 18 \cdot c \cdot 2^{2(k-1)}$$

$$\le 7^k MM(0) + 18c \cdot \left[2^{2(k-1)} + 7 \cdot 2^{2(k-2)} + \dots 7^{k-1} \cdot 1 \right]$$

$$= O(7^k) = O(n^{\log 7})$$

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$$MM(2^k) \leq 7 \cdot MM(2^{k-1}) + 18 \cdot c \cdot 2^{2k-2}$$

- Could also use Master theorem to get $MM(n) = O(n^{\log 7}) \approx O(n^{2.807})$

much better asymptotically
than n^3

Matrix Multiplication Exponent

- We can define ω (or ω_{mult}) as the *matrix multiplication exponent*.
  1. If an algorithm for $n \times n$ matrix multiplication has running time $O(n^{\alpha})$, then $\omega \leq \alpha$.
  2. For any $\varepsilon > 0$, there is an algorithm for $n \times n$ matrix multiplication running in time $O(n^{\omega+\varepsilon})$

intuition: ω is the exponent of best algorithm for matrix multiplication.

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- As we will see today, ω is a fundamental constant in computer science!
- Currently we know $2 \leq \omega < \boxed{2.376}$ may be better today

Open Question

What is the right value of ω?

Historical Remarks

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- Motivated work on better algorithms for all other linear algebraic problems
- introduced complexity of computation of *bilinear functions* and the study of complexity of tensor decompositions

- Administrivia
- Matrix Multiplication
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The Exponent of Linear Algebra

- We just saw how to multiply matrices faster than the naive algorithm
- We also learned about $\omega_{mult} := \omega$
- How fundamental is the exponent of matrix multiplication?

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$$\omega_{determinant}, \quad \omega_{inverse}, \quad \omega_{linear\ \ system}, \quad \omega_{characteristic\ \ polynomial}$$

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$$\omega = \omega_{inverse} = \omega_{determinant}$$

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- More generally, all of these ω_P's are related to ω!

  Matrix multiplication exponent fundamental to linear algebra!

- Administrivia
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- **Matrix Inversion**
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Matrix inverse vs matrix multiplication

- Matrix inverse is at least as hard as matrix multiplication
- How to prove this? *reductions!*

If we can invert matrices quickly, then we can multiply two matrices quickly.

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- Suppose we had an algorithm for inverting matrices
- Consider

$$M \cancel{A} = \begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$$

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$$M = \begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}$$

- Then

$$M^{-1} = \begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix}$$

$$\begin{pmatrix} I & A & 0 \\ 0 & I & B \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} I & -A & AB \\ 0 & I & -B \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

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- So if we could invert in time T, then we can multiply two matrices in time $O(T)$.

invert in $O(n^\alpha)$ ⇒ multiply in $O(n^\alpha)$

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- Let $n = 2^k$, divide matrix M into blocks of size $n/2$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

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$$M = \begin{pmatrix} \underline{A} & B \\ C & D \end{pmatrix}$$

- The inverse of M in block form is given by:

$$M^{-1} = \begin{pmatrix} I & -A^{-1}BS^{-1} \\ 0 & S^{-1} \end{pmatrix} \cdot \begin{pmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{pmatrix}$$

Assuming A and $\underbrace{S := D - CA^{-1}}_{\text{Schur complement}}B$ are invertible

Matrix Multiplication vs Matrix Inversion

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  Assuming A and $S := D - CA^{-1}B$ are invertible
- How do we compute this? *Schur Complement*

  Similar to how we would invert regular matrices! Just pay attention to non-commutativity.

Computing Inverse of Block Matrices

$$\underline{\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ \underbrace{O}_{\text{make this diagonal}} & S \end{pmatrix}$$

$$-CA^{-1}\cdot A + I\cdot C = C - C = 0$$

$$\boxed{-CA^{-1}B + D = S}$$

$$\begin{pmatrix} A & B \\ O & S \end{pmatrix} = \begin{pmatrix} A & \\ & I \end{pmatrix} \begin{pmatrix} I & A^{-1}BS^{-1} \\ O & I \end{pmatrix} \begin{pmatrix} I & 0 \\ O & S \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ O & S \end{pmatrix}^{-1} \qquad \begin{pmatrix} A^{-1} & \\ & I \end{pmatrix} \begin{pmatrix} I & -A^{-1}BS^{-1} \\ O & I \end{pmatrix} \begin{pmatrix} I & 0 \\ O & S^{-1} \end{pmatrix}$$

Computing Inverse of Block Matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & S \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & \\ & I \end{pmatrix} \begin{pmatrix} I & A^{-1}BS^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix}$$

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Runtime Analysis

- The inverse of M in block form is given by:

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- To invert M, we needed to:
  - Invert A
  - Compute $S := D - CA^{-1}B$ $\frac{n}{2} \times \frac{n}{2}$ matrix multiplication

    adition

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- To invert M, we needed to:
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  - Compute $S := D - CA^{-1}B$
  - Invert S

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- Recurrence relation:

$$\underbrace{I(n)}_{\text{\# ops to invert nxn matrix}} \leq 2 \cdot \overbrace{I(n/2)}^{A, S \text{ invert}} + \overset{\text{contant}}{C} \cdot (n/2)^{\omega}$$

Solving Recurrence

- Recurrence relation:

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$$C \cdot \sum_{t=1}^{k} 2^{\omega(k-1-t)} \cdot 2^{t}$$

- Thus

$$\boxed{I(n)} = I(2^k) \leq 2^k \cdot I(1) + C \cdot \sum_{j=0}^{k-1} 2^{\omega j}$$

$$\leq C' \cdot \left(2^k + \frac{2^{\omega k} - 1}{2^{\omega} - 1}\right)$$

$$\leq C'' \cdot 2^{\omega k} = \boxed{C'' n^{\omega}}$$

$\Rightarrow \quad \omega_{inv} \leq \omega$

with previous result $\omega_{inv} \geq \omega \quad \therefore \omega = \omega_{inv}$

$$\underbrace{C \cdot \sum_{t=0}^{k-1} 2^{\omega(k-1-t)+t}}_{\leq C' \cdot \sum_{j=0}^{k-1} 2^{\omega j}} = C \cdot \sum_{j=0}^{k-1} 2^{\omega j + k-1-j}$$

$$j = k-1-t$$

$$= C \cdot 2^{(k-1)} \cdot \underbrace{\sum_{j=0}^{k-1} 2^{(\omega-1)j}}_{\sum_{j=0}^{k-1} \left(2^{\omega-1}\right)^{j}}$$

$$= C \cdot 2^{(k-1)} \cdot \frac{\overset{n^{\omega-1}}{\boxed{\left(2^{\omega-1}\right)^{k} - 1}}}{\underset{\omega \geq 2}{2^{\omega-1} - 1} \neq 0}$$

$$= \frac{C}{2} \cdot n \cdot n^{\omega-1}$$

$$= \frac{C}{\cdots} n^{\omega}$$

Determinant vs Matrix Multiplication

- One can similarly prove that $\omega_{determinant} \leq \omega$
- This is your homework! :)

- Administrivia
- Matrix Multiplication
- The Exponent of Linear Algebra
- Matrix Inversion
- **Determinant and Matrix Inverse**
- Conclusion
- Computing Partial Derivatives

Determinant of a Matrix

- Given matrix $M \in \mathbb{F}^{n\times n}$, the determinant is

$$\det(M) = \sum_{\sigma\in S_n} \underbrace{(-1)^{\sigma}}_{\text{sign}(\sigma)} \cdot \prod_{i=1}^{n} M_{i\sigma(i)}$$

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Remove i^{th} row and j^{th} column of M

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

(2,3)-minor

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- Determinant has a very special decomposition by minors: given any row i, we have

$$\det(M) = \sum_{j=1}^{n} (-1)^{i+j} \underline{M_{i,j}} \cdot \underline{\det(M^{(i,j)})}$$

known as *Laplace Expansion*

↑
none of these depend on any M_{ik}

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- Determinants of minors are very much related to *derivatives* of the determinant of M

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$$\underset{\overset{\uparrow}{(i,j)}}{N_{i,j}} = (-1)^{i+j}\underbrace{\det(M^{(j,i)})}_{(j,i)\ \text{minor}}$$

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- Note that

$$MN = \underline{\det(M)} \cdot I$$

∴ inverse $M^{-1} = \frac{1}{\det(M)} \cdot N$

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$$\det(M^{(i,j)}) = (-1)^{i+j} \partial_{i,j} \det(M)$$

- So, if we knew how to compute the determinant AND ALL its partial derivatives, we could:
  1. Compute the adjugate
  2. Compute the inverse

if can compute det in time n^α ops.

Can we also compute $\det(n)$ and ALL in $O(n^\alpha)$?

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- Suppose we have an algorithm which computes the determinant in $O(n^{\alpha})$ operations

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Computing the Determinant

$$\omega_{det} \geq \omega_{inv} \quad .$$

- Suppose we have an algorithm which computes the determinant in $O(n^{\alpha})$ operations
- Can compute the determinant and all its partial derivatives in $O(n^{\alpha})$ operations!
- Compute the inverse by simply dividing $\det(M^{(i,j)}) / \det(M)$

$$N_{ij} = \left(\partial_{ji} \det(M) \right)$$

we can compute det in time $O(n^{\alpha})$

then can compute inverse $O(n^{\alpha})$

Conclusion

- Today we learned how fundamental matrix multiplication is in symbolic computation and linear algebra
- Used fast computation of partial derivatives to compute the inverse from the determinant

- Administrivia
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Algebraic Circuits - base ring R

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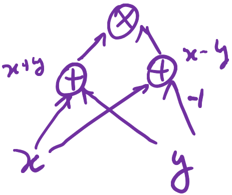
$x^2 - y^2$



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- *circuit size:* number of edges in the circuit, denoted by $\mathcal{S}(\Phi)$

Partial Derivatives

- if $f(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ the partial derivatives
$$\partial_1 f,\ \partial_2 f, \dots,\ \partial_n f$$
are such that
$$\partial_i x_j^d = \begin{cases} d x_j^{d-1}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$
and
$$\partial_i f$$
is computed as above considering all other variables "constant"

$$\partial_i f := \partial_{x_i} f$$

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- Example: $f(x_1, x_2) = x_1^2 x_2 - x_1 x_2^3$

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- How fast can we compute partial derivatives?

if have circuit of size s computes f

what is smallest size of ckt computing all $\partial_i f$?

Computing Partial Derivatives

- If f can be computed using L operations $+, -, \times$, then we can compute *ALL* partial derivatives *simultaneously*

$$\partial_1 f, \ldots, \partial_n f$$

  performing $4L$ operations!

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- Algorithm we will see today discovered independently in Machine Learning - known as *backpropagation*

Computing Partial Derivatives

- We are going to use the chain rule:

$$\underline{\partial_i} f(\underset{y_1}{g_1}, \underset{y_2}{g_2}, \ldots, \underset{y_m}{g_m}) = \sum_{\underline{j=1}}^{m} \underset{y_j}{\underline{(\partial_j f)}}(g_1, g_2, \ldots, g_m) \cdot \underline{\partial_{\bigcirc\!\!\!\!i\;} g_j}$$

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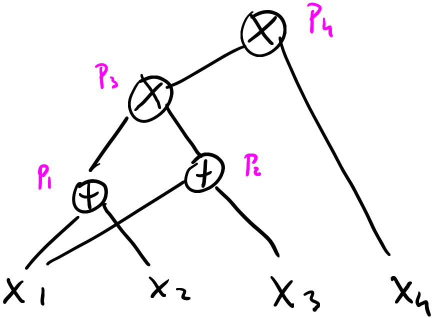
  3. Have to compute partial derivatives "*in reverse*"

Example

- Consider the following computation:

$$P_1 = x_1 + x_2,\ P_2 = x_1 + x_3,\ P_3 = P_1 \cdot P_2,\ P_4 = x_4 \cdot P_3$$



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Computation	∂_1	∂_2	∂_3	∂_4
$P_1 = x_1 + x_2$	1	1	0	0
$P_2 = x_1 + x_3$	1	0	1	0
→ $P_3 = P_1 P_2$	$\underline{P_2} \cdot \underline{\partial_1 P_1} + \underline{P_1} \cdot \underline{\partial_1 P_2}$	$P_2 \cdot \partial_2 P_1$	$P_1 \cdot \partial_3 P_2$	0
$P_4 = x_4 P_3$	$x_4 \cdot \partial_1 P_3$	$x_4 \cdot \partial_2 P_3$	$x_4 \cdot \partial_3 P_3$	P_3

lots of new operations needed
if bottom-up

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- Now let's see how to "do it in reverse"

Example - reverse mode

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Example - reverse mode

- Consider the computation:

replaced by y

$$\boxed{P_1 = x_1 + x_2,}\; P_2 = x_1 + x_3,\; P_3 = P_1 \cdot P_2,\; P_4 = x_4 \cdot P_3$$

- Replacing first computation with a new variable y, we get:

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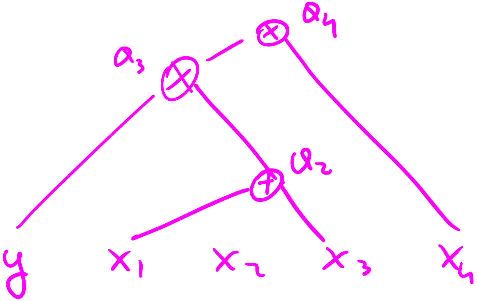

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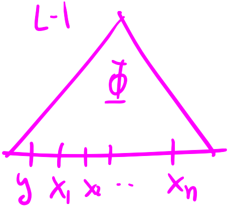

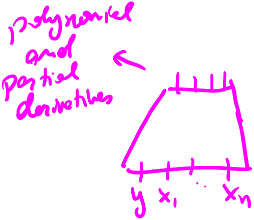

$4(L-1)$

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- Can transform the circuit above into one that computes all partial derivatives of P_4 by using the *chain rule*!
- Note that

$$Q_4(x_1, x_2, x_3, x_4, y = P_1) = P_4$$

Computing Partial Derivatives - Proof

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- By chain rule, we have $1 \leq i \leq 4$

$$\underline{\partial_i Q_4} = \sum_{j=1}^{4} (\partial_j Q_4)(x_1, x_2, x_3, x_4, P_1) \boxed{\cdot (\partial_i x_j)}$$
$$+ (\underline{\partial_y Q_4})(x_1, x_2, x_3, x_4, P_1) \cdot (\partial_i P_1)$$

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$$\begin{aligned}\partial_i P_4 \quad \cancel{\partial_i Q_4} = & (\partial_i Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot 1 \\ & \underset{\nwarrow \text{ new operation}}{\boxed{+}} (\partial_y Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot (\partial_i P_1)\end{aligned}$$

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$\neq 0$ for at most 2 values of i

- *Crucial remark*: note that P_1 depends on at most 2 variables!!

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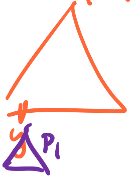
P depends on x_1, x_2

$$\left.\begin{aligned}\partial_3 P_4 &= \underline{\partial_3 Q_4}(\bar{x}, P_1) \\ \partial_4 P_4 &= \partial_4 Q_4(\bar{x}, P_1)\end{aligned}\right\} \text{were already computed!}$$

$\partial_4 Q_4(\bar{x}, y)$



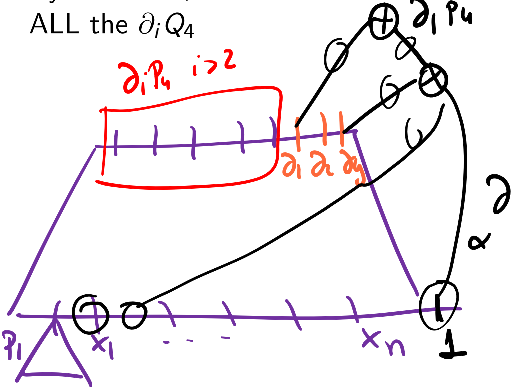
only derivatives left are $\partial_1 P_4$ and $\partial_2 P_4$

Computing Partial Derivatives - Proof

- By chain rule, we have $1 \le i \le 4$

$$\boxed{\begin{aligned}\partial_i P_4 = &(\partial_i Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot 1\\ &+ (\partial_y Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot \underline{(\partial_i P_1)}\end{aligned}}$$

- *Crucial remark*: note that P_1 depends on at most 2 variables!
- By induction, we know a circuit of size $\le 8(L-1)$ which computes ALL the $\partial_i Q_4$



$\partial_i P_4 =$

$\boxed{P_1 = x_1 x_2}$

or

$P_1 = \alpha x_1 + \beta x_2$

$\partial_1 P_1$ or $\partial_2 P_1$ are computed already

Computing Partial Derivatives - Proof

- By chain rule, we have $1 \leq i \leq 4$

$$\begin{aligned}\partial_i Q_4 =&(\partial_i Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot 1 \\ &+ (\partial_y Q_4)(x_1, x_2, x_3, x_4, P_1) \cdot (\partial_i P_1)\end{aligned}$$

- *Crucial remark*: note that P_1 depends on at most 2 variables!
- By induction, we know a circuit of size $\leq 4(L-1)$ which computes ALL the $\partial_i Q_4$
- P_1 is of the form

$$\alpha x_i + \beta x_j, \quad x_i x_j, \quad \alpha x_i + \beta$$

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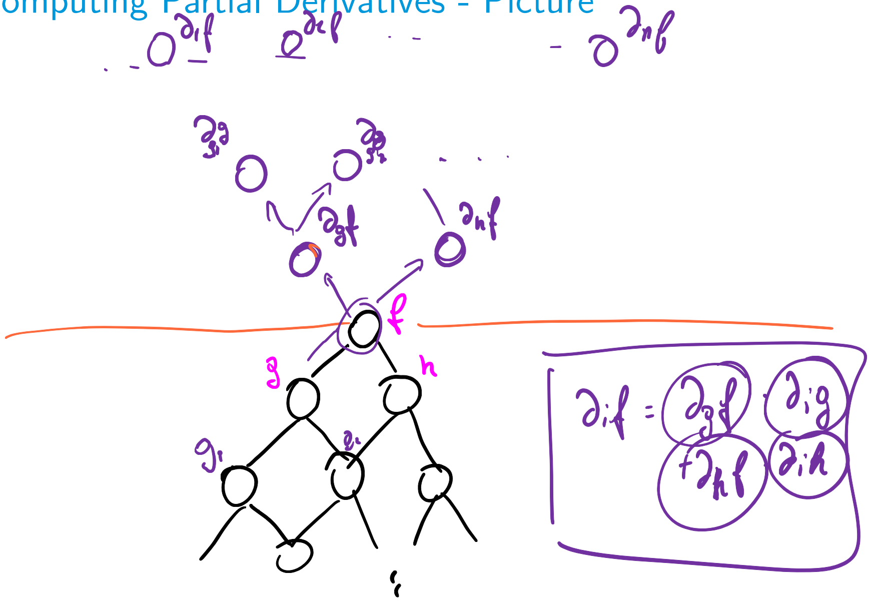
- So circuit computing ALL $\partial_i P_4$ derivatives has size

$$\leq 4(L-1) + 4 = 4L$$

Computing Partial Derivatives - Picture



$$\partial_i f = \partial_g f \, \partial_i g + \partial_h f \, \partial_i h$$