

Lecture 20: Hardness of Approximation

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Overview

- Background and Motivation
 - Why Hardness of Approximation?
 - How do we prove Hardness of Approximation?
 - Hardness of Approximation - Example
- Proofs & Hardness of Approximation
- Conclusion
- Acknowledgements

Why Study Hardness of Approximation?

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Hardness of Approximation

- Important to know the limits of efficient algorithms!

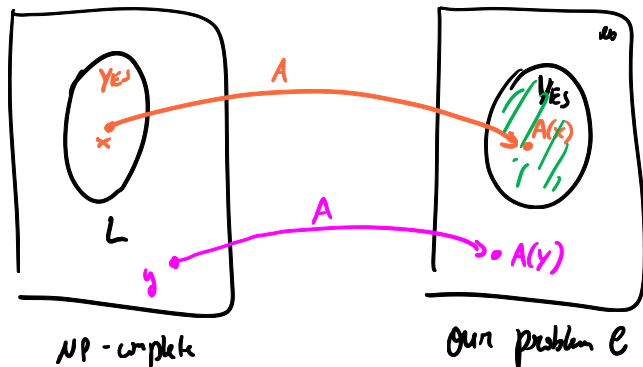
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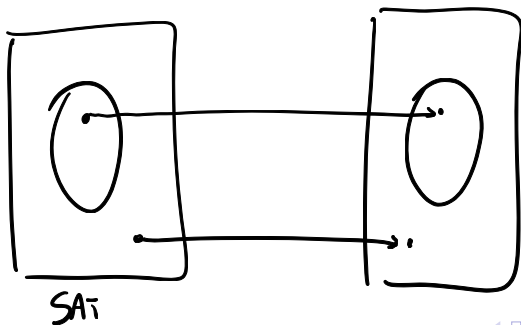
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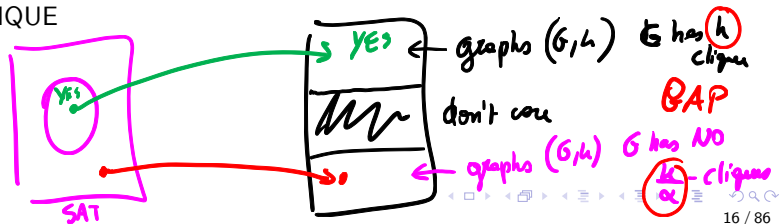
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$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

$$d(a, b) = d(b, a)$$

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 - Efficient route planning (mail system, shuttle bus pick up and drop off...)
- One of the famous NP-complete problems

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 - More generally, if there is any function $r : \mathbb{N} \rightarrow \mathbb{N}$ such that $r(n)$ computable in polynomial time, then it is hard to $r(n)$ -approximate General TSP-NR if we assume that $P \neq NP$

$$r(n) = 2^n$$

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- 3 In our case, let's reduce it to the *Hamiltonian Cycle Problem*

Theorem

If there is an algorithm M which solves TSP without repetitions with α -approximation, then $P = NP$.

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 - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)
 - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \text{ } \textit{small weight for edges in } E \\ (1 + \alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \text{ } \textit{large weight for edges not in } E \end{cases}$

Why is $H(V, F, w)$ an instance of TSP ?

$H(V, F)$ complete graph

$w: V \times V \rightarrow \mathbb{R}_{\geq 0}$ $\left\{ \begin{array}{l} \text{valid} \\ \text{input to} \\ \text{TSP} \end{array} \right.$

symmetric

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- 3 If G has a Hamiltonian Cycle, then OPT for the TSP is of value $\leq |V|$

Ham. Cycle in G gives a cycle of distance $\leq |V|$
in $H(V, F, w)$
valid solution to TSP

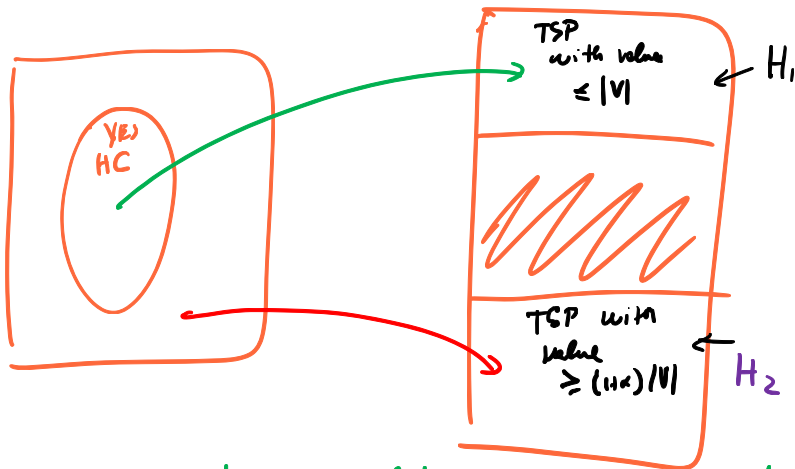
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- 5 Thus, M on input H will output a Hamiltonian Cycle of G , if G has one, or it will output a solution with value $\geq (1 + \alpha) \cdot |V|$

Discussion of Proof



$M(H_1) \leftarrow$ return a solution with value $\leq \alpha|V|$

$M(H_2) \leftarrow$ return a solution with value $\geq \underline{(1+\alpha)|V|}$

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Complexity Classes

- **NP:** Set of languages $L \subseteq \{0, 1\}^*$ such that there exists a poly-time Turing Machine V , such that:

$$x \in L \Leftrightarrow \exists w \in \{0, 1\}^{\text{poly}(|x|)} \text{ s.t. } V(x, w) = 1$$

witness proof

small

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Randomized algorithms

$$\Pr_{R \in \{0, 1\}^{\text{poly}(|x|)}} [M(x, R) = L(x)] \geq 2/3$$

uniformly at random

randomized algo.

$\rightarrow = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{o.w.} \end{cases}$

for most choices of random strings it will output right answer

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PIT \in coRP

coPIT : P is P \neq 0

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- **co-RP:** languages $L \subseteq \{0, 1\}^*$ s.t. $\bar{L} \in RP$

Examples of Problems in Complexity Classes

Matching $\in P$

Ham cycle $\in NP$

$V \left(\underbrace{G(V, E)}_{\text{input}}, \underbrace{\text{cycle } \mathcal{C}}_{\text{witness/proof}} \right) = \text{checks that } \mathcal{C} \text{ is valid Ham. cycle}$

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NP

prover : must give proofs of poly-size (input)

verifier : using algorithms from P

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 - Verifier picks a poly-time Turing Machine V and outputs
$$\begin{cases} TRUE, & \text{if } V(x, w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$$

NP as Proof System - Example

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- 1 Given language L (the language of correct statements)
- 2 $x \in L \Rightarrow$ there exists proof w such that $\Pr[V^w(x) = 1] = 1$

$V^w(x)$ ← oracle calls

Probabilistic Proof Systems

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Definition (Probabilistic Proof System)

In a probabilistic proof system, the verifier has a randomized algorithm V for which:

- 1 Given language L (the language of correct statements)
- 2 $x \in L \Rightarrow$ there exists proof w such that $\Pr[V^w(x) = 1] = 1$
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Definition (Interactive Proof Systems)

The class IP consists of all languages L that have an interactive proof system (P, V) where

- 1 the verifier V is a randomized, polynomial time algorithm
- 2 there is an honest prover P (who can be all powerful)

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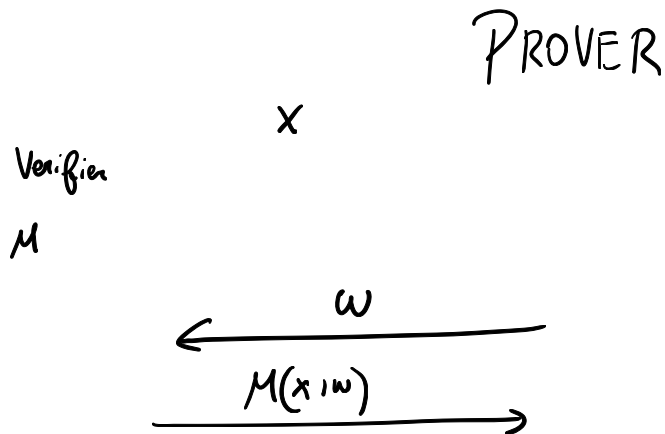
- 1 the verifier V is a randomized, polynomial time algorithm
- 2 there is an honest prover P (who can be all powerful)
- 3 for any $x \in \{0, 1\}^*$
 - $x \in L \Rightarrow$ for an *honest* prover P , the proof Π_P satisfies:

$$\Pr[V^{\Pi_P}(x) = 1] = 1$$

- $x \notin L \Rightarrow$ for *any prover* P' , the proof $\Pi_{P'}$

$$\Pr[V^{\Pi_{P'}}(x) = 1] \leq 1/2$$

Interactive Proofs - Picture



Quantifying Probabilistic Proof Systems

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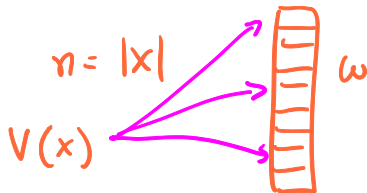
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Verification

- 1 Uses $O(r(n))$ random bits
- 2 Examines $O(q(n))$ bits of a proof w

$r: \mathbb{N} \rightarrow \mathbb{N}$ random bits
 $q: \mathbb{N} \rightarrow \mathbb{N}$ # queries to proof.

Note that n *does not* depend on w , only on x .



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Theorem (PCP theorem [AS'98, ALMSS'98])

$$PCP[\log n, 1] = NP$$

PCP and Approximability of Max 3SAT

Definition (Max 3SAT)

- **Input:** a 3CNF formula φ on boolean variables x_1, \dots, x_n and m clauses $\varphi = (x_1 \vee \neg x_2 \vee x_5) \wedge (x_3 \vee \neg x_7 \vee x_9) \wedge \dots$
- **Output:** the maximum number of clauses of φ which can be simultaneously satisfied.

Theorem

- 1 *The PCP theorem implies that there is an $\varepsilon > 0$ such that there is no polynomial time $(1 + \varepsilon)$ -approximation algorithm for Max 3SAT, unless $P = NP$.*
 - 2 *Moreover, if Max 3SAT is hard to approximate within a factor of $(1 + \varepsilon)$, then the PCP theorem holds.*
- In other words, the PCP theorem and the hardness of approximation of Max 3SAT are equivalent.

PCP and Approximability of Max 3SAT

- 1 Let us assume the PCP theorem holds.
 - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
 - Let V be the $(O(\log n), q)$ verifier for L , where q is a constant

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- 3 Given an instance x of problem L , we construct 3CNF formula φ_x with m clauses such that, for some ε we have
 - $x \in L \Rightarrow \varphi_x$ is satisfiable
 - $x \notin L \Rightarrow$ no assignment satisfies more than $(1 - \varepsilon) \cdot m$ clauses of φ_x

PCP and Approximability of Max 3SAT

$$V(x, R)$$

↑
random bits

- ① Let us assume the PCP theorem holds.
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- ② We now describe a reduction from L to Max 3SAT which has a gap.
- ③ Given an instance x of problem L , we construct 3CNF formula φ_x with m clauses such that, for some ε we have
 - $x \in L \Rightarrow \varphi_x$ is satisfiable
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- ④ Enumerate all random inputs R for the verifier V .
 - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
 - For each R , V chooses q positions i_1^R, \dots, i_q^R and a boolean function $f_R : \{0, 1\}^q \rightarrow \{0, 1\}$ and accepts iff $f_R(w_{i_1^R}, \dots, w_{i_q^R}) = 1$.

$$2^{c \log n} = n^c$$

$$2^{O(\log n)} = \text{poly}(n)$$

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 - Can be done with a CNF of size 2^q
 - Converting to 3CNF we get a formula of size $q \cdot 2^q$

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- 3 Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above

$\text{poly}(n) \cdot q \cdot 2^q$ clauses

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- 4 If $x \in L$ then there is a witness w such that $V(x, w)$ accepts for every random string R . In this case, φ_x is satisfiable!
- 5 If $x \notin L$ then the verifier says NO for half of the random strings R .
 - For each such random string, at least one of its clauses fails $\frac{1}{q \cdot 2^q}$
 - Thus at least $\varepsilon = \frac{1}{2 \cdot q \cdot 2^q}$ of the clauses of φ_x fails.

Digested Proof of Theorem

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Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems
- Proof systems, in particular *Probabilistic Checkable Proofs*, allows us to get such strong reductions
- Many more applications in computer science and industry!
 - Program Checking (for software engineering)
 - Zero-knowledge proofs in cryptocurrencies
 - many more...

Acknowledgement

- Lecture based largely on:
 - Section's 1-3 of Luca's survey [Trevisan 2004]
 - [Motwani & Raghavan 2007, Chapter 7]
- See Luca's survey <https://arxiv.org/pdf/cs/0409043>

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