Lecture 17: Online Algorithms & Paging

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

- Part I
  - Why Study Online Algorithms?
  - Competitive Analysis
  - Examples
- Paging & Caching
- Conclusion
- Acknowledgements

Why Study Online Algorithms?

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.

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- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...

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- Applications in
  - Stock Market
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  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future* (that is, see the entire input beforehand)[1]
  1. Worst-case analysis

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  2. worst-case analysis
- Today, we will only see algorithms which must deal with the input as it receives it, *no constraints in memory*.
  1. Goal here is to *be competitive* against *any offline algorithm* (that is, algorithms that could see the entire input beforehand)
  2. worst-case analysis

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Definition (Deterministic Competitive Ratio)

A deterministic online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

$$C_A(s) \leq k \cdot C_{opt}(s) + O(1)$$

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Definition (Randomized Competitive Ratio)

A randomized online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

$$\mathbb{E}[C_A(s)] \leq k \cdot C_{opt}(s).$$

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- Algorithm has to decide *when to buy*, knowing only that we have gone skiing t times

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  - If $t \geq 10$, we buy at the 10^{th} time, so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot 9 + 1000}{1000} = 1.9$$

$C_A(t) \leq 1.9\, C_{opt}(t)$

$\forall$ input sequence t = # times we go skiing in entire lives

~~Secretary~~ Dating Problem

- In the high-tech life, you decide to join a dating site...

[3] Assumptions: people are comparable AND we know how to do it

[4] Go big or go ~~home~~ lonely!

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- **Goal:** maximize probability of dating the best person

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- Suppose we pick a person at time k, then want to compute probability

$$P_k = \Pr[\pi(k) = 1 \text{ and we pick person at time } k]$$

1st person shows up at time k

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- If $\pi(k) = 1$, then $1 - P_k$ is the probability that we picked a person between $[t+1, k-1]$, which means someone in this range better than the first t people.

ranks of all dates up to date k-1

$$P_k = \Pr[\pi(k) = 1 \text{ and } \min \pi(1), \ldots, \pi(k-1) \text{ is in } \{\pi(1), \ldots, \pi(t)\}]$$

we did not pick anyone before date k

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$$= \frac{1}{n} \cdot \frac{t}{k-1}$$

picking a permutation of $a_1 < a_2 < \cdots < a_{k-1}$ s.t. a_1 appears within first t places

$$\frac{t}{k-1}$$

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$$\begin{aligned} P_k &= \Pr[\pi(k) = 1 \text{ and } \min \pi(1), \ldots, \pi(k-1) \text{ is in } \{\pi(1), \ldots, \pi(t)\}] \\ &= \frac{1}{n} \cdot \frac{t}{k-1} \end{aligned}$$

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$$P = \sum_{k>t}^{n} \frac{1}{n} \cdot \frac{t}{k-1} = \frac{t}{n} \cdot \sum_{k>t}^{n} \frac{1}{k-1} \approx \frac{t}{n} \cdot (\ln n - \ln t) = \frac{t}{n} \cdot \ln(n/t)$$

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- Optimizing we get that we should set $t = n/e$, which gives us $1/e$ probability.
- Wait a second, where is the competitive analysis?

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input : sequence of people

$p_1 \; p_2 \; \cdots \; p_n$

$\pi(p_1) \; \pi(p_2) \; \cdots \; \pi(p_n)$

$\pi : [n] \to [n]$ this exists but don't know

all we can do is $comp(p_i, p_j) \to \arg\min(\pi(p_i), \pi(p_j))$

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  - We can simply minimize the rank.
  - Say we always want to end up with someone (loneliness has a cost of $-\infty$, after all nobody wants to be alone)
  - Previous algorithm would then either pick the best person, or the last person in the order.
  - With constant probability, rank of the last person is $\Omega(n)$, so we either date the best, or we date someone in the "bottom percentile" of our list

$$\mathbb{E}[C_A(\sigma)] = \Omega(n)$$

$$\geq \underbrace{\Pr[\text{end up last person}]}_{\text{constant}} \cdot \underbrace{(\text{rank of last person})}_{\Omega(n)}$$

Making Dating Competitive

- To make the dating problem competitive, we would have to modify it a little bit.
  - We can simply minimize the rank.
  - Say we always want to end up with someone (loneliness has a cost of $-\infty$, after all nobody wants to be alone)
  - Previous algorithm would then either pick the best person, or the last person in the order.
  - With constant probability, rank of the last person is $\Omega(n)$, so we either date the best, or we date someone in the "bottom percentile" of our list
  - Expected rank of our life-long partner is $\Omega(n)$
- What is $C_{opt}(s) = 1$

  opt can always see in the future

  (in particular opt knows rank function)

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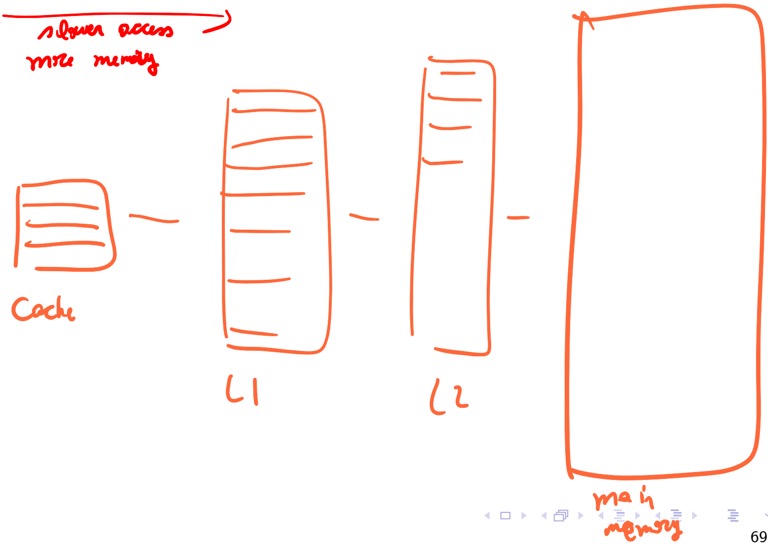
- That is, as we get older, we become more desperate to find someone and lower our expectations...

Online Paging Problem

- Computer memory is hierarchical: cache → L1 → L2 → main memory



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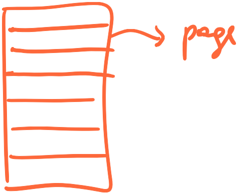

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thing that takes most time is fetching data from slower memory

→ also have to write the new datum in cache

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- Simplification: assume we only have cache and main memory.

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1. **Least Recently Used (LRU)**: delete page in cache whose *most recent request* happened *furthest* in the past

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Today, we will analyze the **Least Recently Used** heuristic. We will assume that *the size of our cache is k pages*.

1. **Least Recently Used (LRU)**: k-competitive
2. **Random**: k-competitive
3. **First-in, First-out (FIFO)**: k-competitive
4. **Least Frequently Used (LFU)**: NOT competitive

LRU Analysis

Theorem

For cache of size k, LRU is k-competitive.

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1. Upper bound: divide input sequence into phases.
   - First phase starts immediately after our algorithm first faults, ends right after the algorithm faults k more times
   - Second phase starts at end of first phase, ends when algorithm faults for additional k times
   - and so on...

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2. We will prove that OPT algorithm faults *at least* once per phase

in each phase: OPT faults ≥ 1

A faults $= k$

$$\frac{C_A(s)}{C_{OPT}(s)} \leq \frac{k}{1}$$

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4. Examples of phases, for $k = 3$:

request → 1, 1, 2, 2, 1, 3, 4, 3, 2, 4, 5, 6, 15, 4, 4, 2, 3, 5, 6, 4,5

sequence for pages

LRU Analysis

assumption: we start with empty cache (any algorithm)

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1 (1, 2, 2, 1, 3, 4) (3, 2, 4, 5, 6) (15, 4, 4, 2) (3, 5, 6) (4, 5

first miss — 3 cache misses

LRU Analysis - Example

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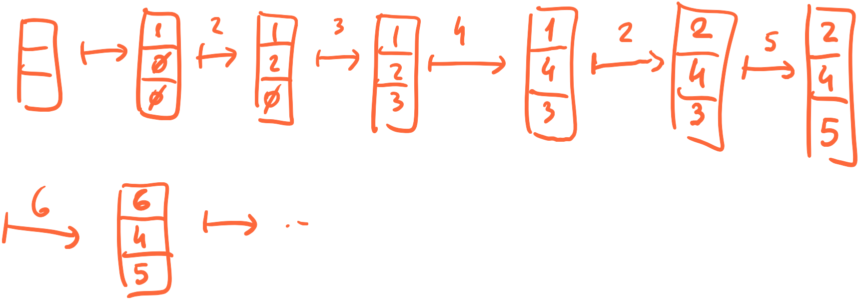

LRU Analysis - Upper Bound

- Need to prove that OPT will fault at least once per phase.
- If the same page faulted twice in one phase:

page p faulted twice in one phase (we faulted when requesting p)

intermediate requests

r_1 r_2 r_3 -- r_{m-1} r_m within a phase

r_1: page p — first req. of p that faulted; had no page p in cache

r_m: page p and faulted again

⇒ must have queried k-1 different pages (∵ p LRU page) (must have evicted p) ← extra page

⇒ must have made k+1 distinct requests in this phase

(k-1 distinct pages after r_1 + page that evicted p + page p)

pigeon hole principle OPT has to fault once

LRU Analysis - Upper Bound

- If each page faulted once in a phase.

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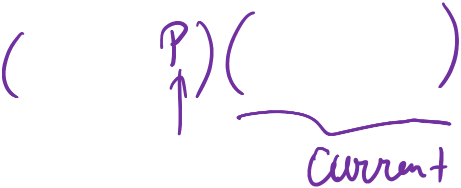
- If each page faulted once in a phase.
- **Claim:** in the beginning of each phase, content of OPT and content of our algorithm A intersect in at least one page.
- Proof: Look at last fault page in previous phase.



A faulted at P so in current phase cache of A has page P.

if OPT did not fault at P OPT already had P in its cache

faulted at P then P will now be in OPT's cache

LRU Analysis - Upper Bound

- If each page faulted once in a phase.
- **Claim:** in the beginning of each phase, content of *OPT* and content of our algorithm *A* intersect in at least one page.

- Since *OPT* and *A* had a common page, then *OPT* must have faulted as well (since each page faulted in this phase)

⇒ have k distinct page faults

⇒ we will have k distinct page requests

OPT could only have already had k-1 out of the k pages that were requested

$P_1, \ldots P_k$ are the faulty requests $P_i \neq P_j$ $i \neq j$ $P \in$ Cache$_{OPT}$

OPT ⟷ $(P_1\ P_2 \ldots P_k)$ in beg. of phase but we know this doesn't happen because and $P \neq P_i$'s

Lower Bound - Deterministic Paging Algorithms

Theorem

Any deterministic algorithm for paging with k pages is at least k-competitive!

- Proof by trolling.[7] Let's use $k + 1$ pages, and let A be our paging algorithm.

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- When offline algorithm deletes a page, it's next delete happens after at least k steps.

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Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*

Acknowledgement

- Lecture based largely on:
  - Lecture 17 of Luca's Optimization class
  - Lectures 19 and 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Luca's Lecture 17 notes at

  `https://lucatrevisan.github.io/teaching/cs261-11/lecture17.pdf`
- See Karger's Lecture 19 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s22-online.pdf`
- See Karger's Lecture 20 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf`

References I

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