Lecture 16: Semidefinite Programming Relaxation and MAX-CUT

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Overview

- Max-Cut SDP Relaxation
- Max-Cut SDP Rounding
- Conclusion
- Acknowledgements

Relax... & Round!

In our quest to get efficient (exact or approximate) algorithms for problems of interest, the following strategy is very useful:

[1] Even more general mathematical program, so long as derive SDP from it.

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3. We are still maximizing the same objective function, but over a (potentially) larger set of solutions.

$$OPT(SDP) \geq OPT(QP)$$

↑ if our QP is maximization problem

because SDP is over a larger domain

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   1. If solution to SDP is *integral* and *one-dimensional*, then it is a solution to QP and we are done
   2. If solution has *higher dimension*, then we have to devise *rounding procedure* that transforms

      high dimensional solutions $\rightarrow$ integral & 1D solutions

      rounded SDP solution value $\geq c \cdot OPT(QP)$

[1] Even more general mathematical program, so long as derive SDP from it.

Max-Cut

Maximum Cut (Max-Cut):

$G(V, E)$ graph.

Cut $S \subseteq V$ and size of cut is

$$|E(S, \overline{S})| = |\{(u, v) \in E \ | \ u \in S, v \notin S\}|.$$

Goal: find cut of maximum size.

edges across the cut

Example - Weighted Variant

Maximum Cut (Max-Cut):

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Cut $S \subseteq V$ and weight of cut is the sum of weights of edges crossing cut.
Goal: find cut of maximum weight.

Max-Cut

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Quadratic Program:

$$\text{maximize} \quad \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - x_u x_v)$$

$$\text{subject to} \quad x_v^2 = 1 \quad \text{for } v \in V$$

value of the cut

$x_v^2 = 1$, $x_v \in \mathbb{R}$ then $x_v \in \{\pm 1\}$

$$1 - x_u x_v = \begin{cases} 2 & \text{if } x_u \neq x_v \\ 0 & \text{otherwise} \end{cases}$$

SDP Relaxation [Delorme, Poljak 1993]

$G(V, E, w)$ weighted graph, $|V| = n$ and $\sum_{e \in E} w_e = 1$

Semidefinite Program:

$$\begin{aligned} \text{maximize} \quad & \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - y_u^T y_v\right) \\ \text{subject to} \quad & \|y_v\|_2^2 = 1 \quad \text{for } v \in V \\ & y_v \in \mathbb{R}^d \quad \text{for } v \in V \end{aligned}$$

$x_v \in \mathbb{R} \longrightarrow y_v \in \mathbb{R}^d$

$x_v^2 = 1 \longrightarrow \|y_v\|^2 = 1$

$x_u x_v \longrightarrow y_u \cdot y_v = y_u^T y_v$

what value of d should we take ?

$d = n$ suffices

$Y = \begin{pmatrix} | & | & & | \\ y_1 & y_2 & \cdots & y_n \\ | & | & & | \end{pmatrix}$

$Y^T Y = X$ ← (X must be PSD)

X n×n symmetric matrix

$X_{ij} = y_i \cdot y_j$

and

$X_{ii} = \|y_i\|^2$

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in unit sphere of $\mathbb{R}^d$

$$\text{maximize} \sum_{i,j \in E} \frac{1}{2} w_{ij} (1 - x_{ij})$$

$$\text{s.t.} \quad x_{ii} = 1 \quad \forall i \in [n]$$

$$\text{and } X \succcurlyeq 0$$

What is this SDP doing?

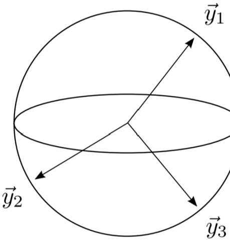

Figure 10.1: Vectors $\vec{y}_v$ embedded onto a unit sphere in $\mathbb{R}^d$.

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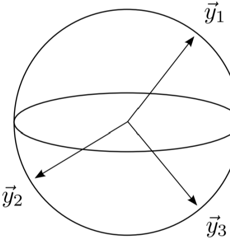

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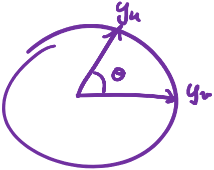


- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$

inner product

$$\langle y_u, y_v \rangle = \|y_u\| \cdot \|y_v\| \cdot \cos(y_u, y_v)$$

$$= 1 \cdot 1 \cdot \cos\theta$$

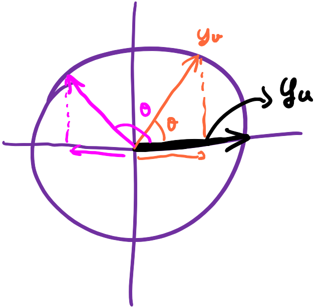
What is this SDP doing?

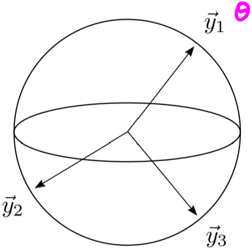
$\theta \in [0, \pi]$

$\theta \to$ small $\leftrightarrow$ large cosine

$\theta > \pi/2 \leftrightarrow$ cosine negative

$\theta = \pi \leftrightarrow$ cosine $= -1$





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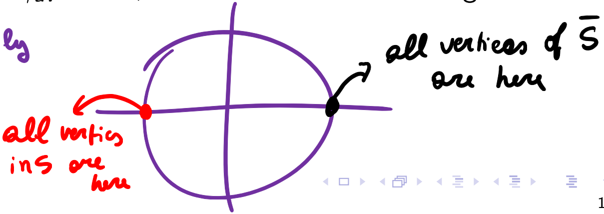
- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight

a cut is exactly



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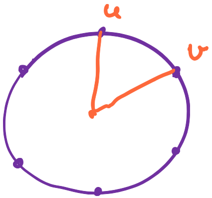


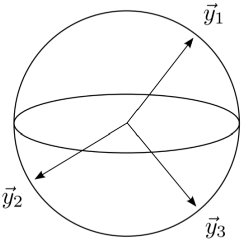
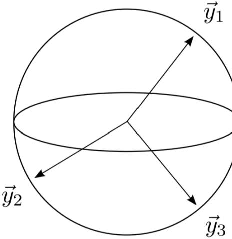


Figure 10.1: Vectors $\vec{y}_v$ embedded onto a unit sphere in $\mathbb{R}^d$.

SDP is pushing apart vertices which have an edge of high weight between them

- Let $\gamma_{u,v} = y_u^T y_v = \cos(y_u, y_v)$
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight
- Geometrically, want vertices from our max-cut S to be as far away from the complement $\overline{S}$ as possible

What is this SDP doing?



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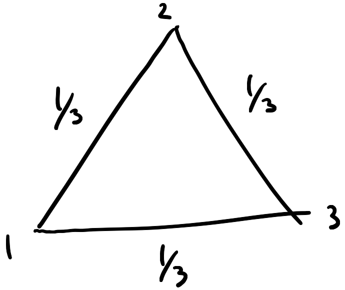
- for any edge, want $\gamma_{uv} \approx -1$, as this maximizes our weight
- Geometrically, want vertices from our max-cut S to be as far away from the complement $\overline{S}$ as possible
- If all y_v's are in a one-dimensional space, then we get original quadratic program

$$OPT(SDP) \geq \text{Weight of Maximum Cut}$$

Example

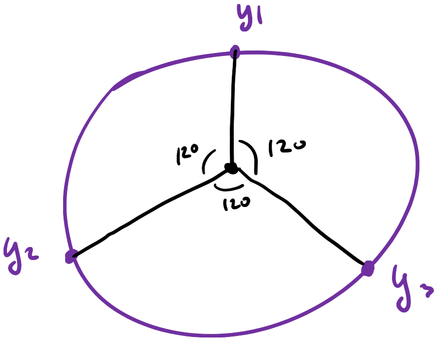
Let's consider $G = K_3$ with equal weights on edges.



Example

Let's consider $G = K_3$ with equal weights on edges.

- Embed $y_1, y_2, y_3 \in \mathbb{R}^2$ 120 degrees apart in unit circle



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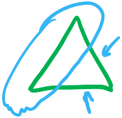
Example

exercise

Let's consider $G = K_3$ with equal weights on edges.

- Embed $y_1, y_2, y_3 \in \mathbb{R}^2$ 120 degrees apart in unit circle
- We get:
- $OPT_{SDP}(K_3) = 3/4$
- $OPT_{\text{max-cut}}(K_3) = 2/3$



$$\max \quad \frac{1}{2} \cdot \frac{1}{2} \cdot \Big[\big(1 - \cos(y_1, y_2)\big) + \big(1 - \cos(y_1, y_3)\big) + \big(1 - \cos(y_2, y_3)\big) \Big]$$

y_i unit sphere in $\mathbb{R}^3$

if not $y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $y_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $y_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$OPT_{SDP} \stackrel{?}{=} \frac{1}{4} \cdot 3 = 3/4$

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- **Practice problem:** try this with C_5.

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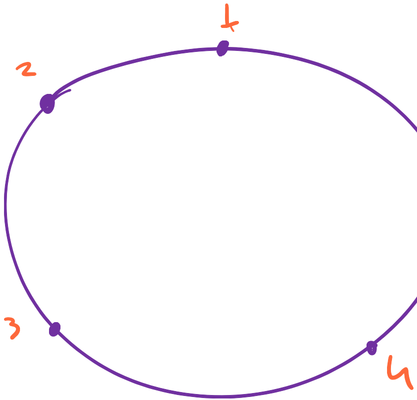
1. Let $z_u \in \mathbb{R}^n$ be an optimal solution to our SDP
2. How do we convert it into a cut?

$\{1,2\} \in E$

$\{1,3\}$



intuitively $w_{13} > w_{12}$

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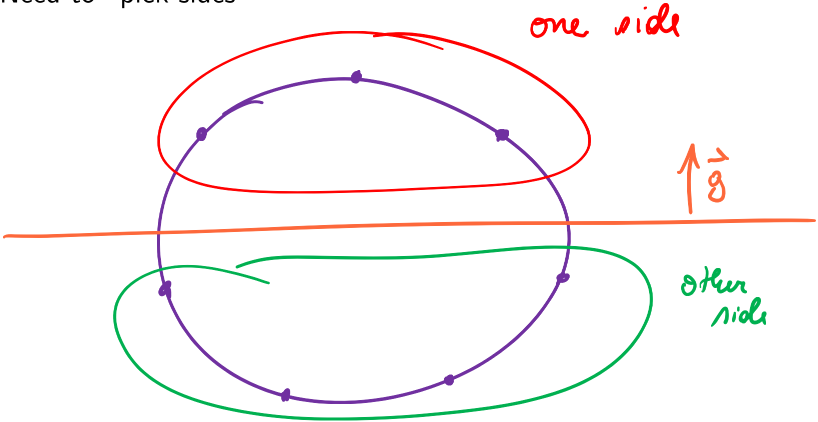
3. Need to "pick sides"



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2. How do we convert it into a cut?
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4. **[Goemans, Williamson 1994]:** Choose a random hyperplane though origin!

hope that SDP separated a lot of edges with high weight
(intuition)

and that a random hyperplane will catch a good cut

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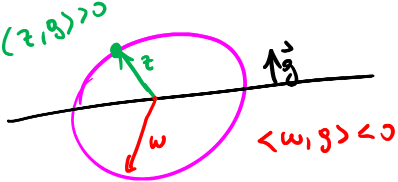
4. **[Goemans, Williamson 1994]:** Choose a random hyperplane though origin!
5. Choose normal vector $g \in \mathbb{R}^n$ from a Gaussian distribution.
6. Set $x_u = \text{sign}(g^T z_u)$ as our solution

$$H = \{ x \in \mathbb{R}^n \mid \langle \vec{g}, x \rangle = 0 \}$$

normal vector



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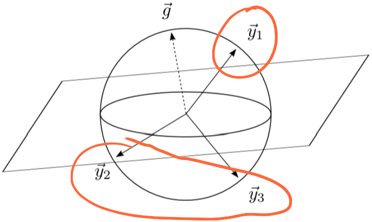


Figure 10.2: Vectors being separated by a hyperplane with normal $\vec{g}$.

Facts we need

- We can pick a random hyperplane through origin in polynomial time.

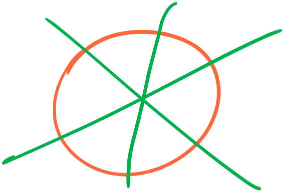
  sample vector $g = (g_1, \ldots, g_n)$ by drawing $g_i \in \mathcal{N}(0,1)$

Normal or Gaussian distribution

Facts we need

- We can pick a random hyperplane through origin in polynomial time.

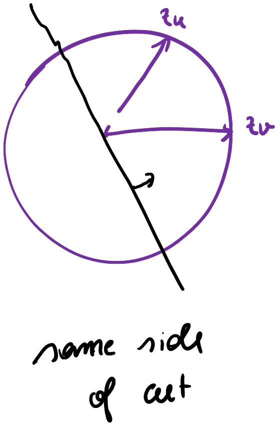
  sample vector $g = (g_1, \ldots, g_n)$ by drawing $g_i \in \mathcal{N}(0, 1)$
- If g' is the projection of g onto a two dimensional plane, then $g'/\|g'\|_2$ is *uniformly distributed* over the unit circle in this plane.

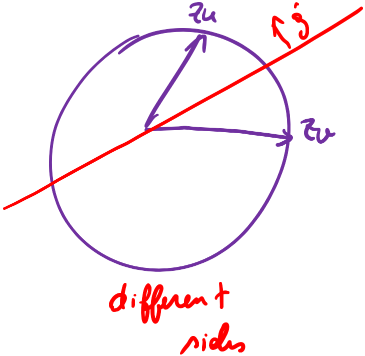


Analysis of Rounding

- Probability that edge $\{u, v\}$ crosses the cut is same as probability that z_u, z_v fall in different sides of hyperplane

$$\Pr[\{u, v\} \text{ crosses cut }] = \Pr[g \text{ splits } z_u, z_v]$$





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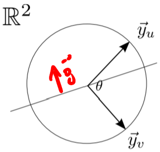
- Looking at the problem in the plane:



hyperplane we choose is random by fact 2

Figure 10.3: The plane of two vectors being cut by the hyperplane

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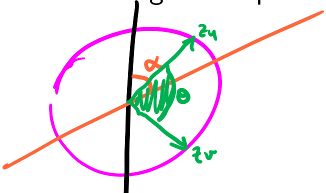

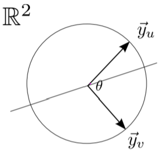

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- Probability of splitting z_u, z_v:

$$\Pr[\{u, v\} \text{ crosses cut}] = \frac{\theta}{\pi} = \frac{\cos^{-1}(z_u^T z_v)}{\pi} = \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

Analysis of Rounding

- Expected value of cut:

$$\mathbb{E}[\text{value of cut}] = \sum_{\{u,v\} \in E} w_{u,v} \cdot \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

weight of edge u,v

Pr[{u,v} is a cut edge]

E[value of u,v]

Analysis of Rounding

- Expected value of cut:

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- Recall that

$$OPT_{SDP} = \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - z_u^T z_v\right) = \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - \gamma_{uv})$$

because $\{z_u\}_{u \in V}$ is an optimum solution

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- If we find α such that

$$\frac{\cos^{-1}(\gamma_{uv})}{\pi} \geq \frac{\alpha}{2}(1 - \gamma_{uv}), \quad \text{for all } \gamma_{uv} \in [-1, 1]$$

Then we have an α-approximation algorithm!

$\Rightarrow$ $\mathbb{E}[\text{value of your cut}] \geq \alpha \cdot OPT_{SDP}$

$\| $

$\sum w_{uv} \cdot \frac{\cos^{-1}(\gamma_{uv})}{\pi} \geq \sum w_{uv} \cdot \frac{\alpha}{2}(1-\gamma_{uv}) = \alpha \cdot \sum \frac{1}{2} w_{uv}(1-\gamma_{uv})$

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- For $x \in [-1, 1]$, we have

$$\frac{\cos^{-1}(x)}{\pi} \geq \underbrace{0.878}_{\alpha =} \cdot \frac{1 - x}{2}$$

  proof by elementary calculus.

Conclusion of rounding algorithm

with constant probability we get a cut
with $(0.878 - \epsilon) \cdot OPT$

(can amplify this probability)

Putting Everything Together

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Putting Everything Together

1. Formulate Max-Cut problem as Quadratic Programming
2. Derive SDP from the quadratic program *SDP relaxation*
3. We are still maximizing the same objective function (weight of cut), but over a (potentially) larger (*higher-dimensional*) set of solutions.

   $$OPT(SDP) \geq OPT(\text{Max-Cut})$$

4. Solve SDP optimally using efficient algorithm.
   1. If solution to SDP is *integral* and *one dimensional*, then it is a solution to Max-Cut and we are done
   2. If have *higher dimensional* solutions, *rounding procedure*

      Random Hyperplane Cut algorithm, ~~with high probability~~ we get

      $$\mathbb{E}[\text{cost(rounded solution)}] \geq 0.878 \cdot OPT(SDP) \geq 0.878 \cdot OPT(\text{Max-Cut})$$

Remarks

1. SDPs are very powerful for solving (approximating) many hard problems

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in particular UGC $\Rightarrow$ cannot obtain better than 0.878 approx. for MAX-cut

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All of these are amazing final project topics!

Conclusion

- Mathematical programming - very general, and pervasive in (combinatorial) algorithmic life
- Mathematical Programming hard in general
- Sometimes can get SDP rounding!
- Solve SDP and round the solution
  - Deterministic rounding when solutions are nice
  - Randomized rounding when things a bit more complicated

Acknowledgement

- Lecture based largely on:
  - Lecture 14 of Anupam Gupta and Ryan O'Donnell's Optimization class

    `https://www.cs.cmu.edu/~anupamg/adv-approx/`
  - Chapter 6 of book [Williamson, Shmoys 2010]
- See their notes at

  `https://www.cs.cmu.edu/~anupamg/adv-approx/lecture14.pdf`

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