Lecture 14: Positive Semidefinite Matrices & Semidefinite Programming

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Overview

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- Application: Control Theory
- Conclusion
- Acknowledgements

Symmetric Matrices & Spectral Theorem

- A matrix $S \in \mathrm{Mat}(n, \mathbb{R})$ is symmetric if $S_{ij} = S_{ji}$ for all $i, j \in [n]$.

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- $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if there exists $u \in \mathbb{C}^n$ such that $Su = \lambda u$. The vector u is an *eigenvector* of S corresponding to λ.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one eigenvalue 1

eigenvector e_1

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ eigenvalues 1, 2

eigenvectors e_1 e_2

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- **Spectral theorem:** any symmetric matrix in $\mathrm{Mat}(n, \mathbb{R})$ has n real eigenvalues (counting with multiplicity), as well as an orthonormal basis (in $\mathbb{R}^n$) for the eigenvectors.

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- **Spectral theorem:** any symmetric matrix in $\mathrm{Mat}(n, \mathbb{R})$ has n real eigenvalues (counting with multiplicity), as well as an orthonormal basis (in $\mathbb{R}^n$) for the eigenvectors.
- In other words, we can write

orthonormal basis for eigenvectors

$$S = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

eigenvalue

where $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$ such that $\boxed{\langle u_i, u_j \rangle = \delta_{ij}.}$

Characterizations of Positive Semidefinite Matrices

- If a symmetric matrix $S \in \mathrm{Mat}(n, \mathbb{R})$ only has non-negative eigenvalues, we say that S is *positive semidefinite* (PSD), and we write $S \succeq 0$.

$\lambda_i \geq 0$

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smallest d
is rank (S)

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  4. $S = LDL^T$, where D is diagonal and non-negative, and L is unit lower-triangular

$$D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} \qquad L = \begin{pmatrix} 1 & & & & 0 \\ * & 1 & & & \\ * & * & 1 & & \\ * & \vdots & * & \ddots & \\ \vdots & & & \ddots & \\ * & \cdots & \cdots & * & 1 \end{pmatrix}$$

$$d_i \geq 0$$

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  5. S is in the convex hull of the set

$$\{uu^T \mid u \in \mathbb{R}^n\}$$

$\alpha_i = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}$ $\quad v_i = \beta_i u_i$

$\beta_i = \sqrt{\sum \lambda_i}$

$$S = \sum_{i=1}^{n} \lambda_i u_i u_i^T \iff S = \sum_{i=1}^{n} \alpha_i v_i v_i^T$$

$\lambda_i \geq 0$ $\qquad \sum_{i=1}^{n} \alpha_i = 1 \quad \alpha_i \geq 0$

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  6. $S = U^T DU$, where D is diagonal and non-negative and $U \in \mathrm{Mat}(n, \mathbb{R})$ is orthonormal matrix (that is, $U^T U = I$).

$$S = \sum_{i=1}^{n} \lambda_i u_i u_i^T \leftrightarrow \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} - u_1^T - \\ - u_2^T - \\ \vdots \end{pmatrix}$$

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  7. Any principal minor of $\cancel{A}$ S has non-negative determinant

$S = \left(\boxed{} \right)$

$S: \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$\{1,3\}$ minor

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

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- **Practice problem:** prove that these are all equivalent!

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Mathematical Programming

Mathematical Programming deals with problems of the form

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_1(x) \geq 0 \\ & \quad \vdots \\ & g_m(x) \geq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

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- Special case when all $f, g_1, \ldots, g_m$ are *linear*. *Linear Programming*

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"positive semidefinite constraints"

g_i will essentially be linear constraints or eigenvalue constraints

$S \succeq 0$

$\Updownarrow$

$\lambda_i(S) \geq 0$

- Very general family of problems.
- Special case when all $f, g_1, \ldots, g_m$ are *linear*. affine *Linear Programming*
- More general case: *Semidefinite Programming*
  1. $A_1, \ldots, A_n, B \in \mathcal{S}^m$ are $m \times m$ symmetric matrices

$\hookrightarrow$ space of all $m \times m$ symmetric matrices

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$$\sum_{i=1}^{n} x_i A_i - B \succeq 0$$

- Very general family of problems.
- Special case when all $f, g_1, \ldots, g_m$ are *linear*. *Linear Programming*
- More general case: *Semidefinite Programming*
  1. $A_1, \ldots, A_n, B \in \mathcal{S}^m$ are $m \times m$ symmetric matrices
  2. Constraints:
     $$x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B$$
  3. Minimize linear function $c^T x$

What is a Semidefinite Program?

- $\mathcal{S}^m := \mathcal{S}^m(\mathbb{R})$ space of all $m \times m$ symmetric matrices (real entries)

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$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\ & x \in \mathbb{R}^n \\ & \boxed{A_i, B \in \mathcal{S}^m(\mathbb{R}) \text{ (fixed matrices)}} \end{array}$$

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Semidefinite Programming deals with problems of the form

$$
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& x \in \mathbb{R}^n \\
& A_i, B \in \mathcal{S}^m(\mathbb{R}) \text{ (fixed matrices)}
\end{array}
$$

Where we use $C \succeq D$ to denote that $C - D \succeq 0$ (i.e., $C - D$ is PSD).

How does it generalize Linear Programming?

Linear Programming

$$\begin{aligned} \text{minimize} \quad & a^T x \\ \text{subject to} \quad & Cx \geq b \\ & x \in \mathbb{R}^n \end{aligned}$$

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$$C = \begin{pmatrix} - & c_1 & - \\ - & c_2 & - \\ & \vdots & \\ - & c_m & - \end{pmatrix}$$

$$c_i x \geq b_i \Leftrightarrow \operatorname{diag}(c_i x - b_i) \succeq 0$$

$$c_i x = \sum_{j=1}^{n} c_{ij} x_j$$

$$\Downarrow$$

$$\sum_{j=1}^{n} A_j x_j - B \succeq 0$$

$$A_j = \begin{pmatrix} c_{1j} & & & 0 \\ & c_{2j} & & \\ 0 & & \ddots & \\ & & & c_{mj} \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_m \end{pmatrix}$$

How does it generalize Linear Programming?

Linear Programming	**Semidefinite Programming**
minimize $a^T x$	minimize $c^T x$
subject to $Cx \geq b$	subject to $x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B$
$x \in \mathbb{R}^n$	$x \in \mathbb{R}^n$

Set A_i's to be diagonal matrices, and $B = diag(b_1, \ldots, b_m)$

$$\sum_{j=1}^{n} A_j x_j - B = \underbrace{\begin{pmatrix} c_1 x - b_1 & & & 0 \\ & c_2 x - b_2 & & \\ 0 & & \ddots & \\ & & & c_m x - b_m \end{pmatrix}}_{\text{eigenvalues } c_i x - b_i} \succeq 0$$

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  - robust optimization
  - statistics and ML
  - continuous games
  - software verification
  - filter design
  - quantum computation and information
  - automated theorem proving
  - packing problems
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  - many more
- See more here

`https://windowsontheory.org/2016/08/27/proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/`

Important Questions

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B \\
& x \in \mathbb{R}^n
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$$

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1. When is a Semidefinite Program *feasible*?
   - Is there a solution to the constraints at all?
   - how do we check quickly that a solution is feasible?

Important Questions

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   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?

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   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?
4. How do we design *efficient algorithms* that find *optimal solutions* to Semidefinite Programs?

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Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

$$\sum A_i x_i \preceq B$$

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Definition (Linear Matrix Inequalities)

A linear matrix inequality is an inequality of the form:

$$A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0,$$

where $A_0, \ldots, A_n$ are *symmetric matrices*.

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Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

one LMI

$$S = \left\{ x \in \mathbb{R}^n \ \middle|\ \sum_{i=1}^{n} A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}$$

Spectrahedra

To understand SDPs, we need to understand their *feasible regions*, which are called *spectrahedra* and described as *Linear Matrix Inequalities* (LMIs).

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \sum A_i x_i \succeq B \\ \sum C_i x_i \succeq D \end{array} \right\} \iff \sum_{i=1}^{n} E_i x_i \succeq F$$

$$\underbrace{E_i = \begin{pmatrix} A_i & 0 \\ 0 & C_i \end{pmatrix} \qquad F = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}}_{\text{symmetric}}$$

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Spectrahedra are <u>convex</u> sets: $x, y \in S$ $\quad z = \alpha x + (1-\alpha) y$

$\alpha \in (0,1)$

$$\sum A_i z_i = \sum A_i (\alpha x_i + (1-\alpha) y_i) = \underbrace{\alpha}_{\geq 0} \underbrace{\sum A_i x_i}_{\succeq B} + \underbrace{(1-\alpha)}_{\geq 0} \underbrace{\sum A_i y_i}_{\succeq B}$$

$$\succeq \alpha B + (1-\alpha) B = B$$

$\Rightarrow z \in S$

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Example of Spectrahedron

Polyhedron:

$P = \{ x \in \mathbb{R}^n \mid Ax \geq b \}$ Linear Programming

$\sum A_{ki} x_i \geq b_k$ k^{th} constraint

$$\underbrace{\sum_{i=1}^{n} \begin{pmatrix} A_{1i} & & & \\ & A_{2i} & & \\ & & \ddots & \\ & & & A_{mi} \end{pmatrix} x_i \succcurlyeq \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_m \end{pmatrix}}_{\text{LMI}}$$

Example of Spectrahedron

Circle:

$$\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \}$$

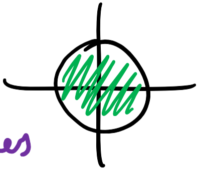
$$\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 \mid \underbrace{\boxed{\begin{pmatrix} \boxed{1+x} & y \\ y & \boxed{1-x} \end{pmatrix}}}_{\text{LMI}} \succcurlyeq 0 \}$$

$$x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \succcurlyeq -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

all determinants of principal minors are ≥ 0

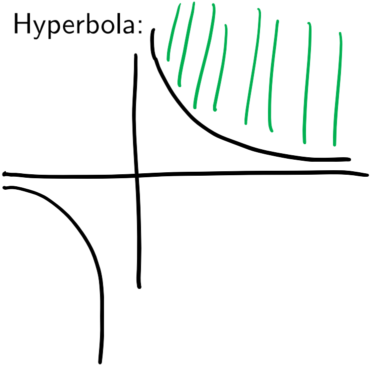
$$\left.\begin{array}{l} 1+x \geq 0 \\ 1-x \geq 0 \end{array}\right\} x^2 \leq 1$$

$$(1+x)(1-x) - y^2 \geq 0 \iff \boxed{1 \geq x^2 + y^2} \text{ suffices}$$



Example of Spectrahedron

Hyperbola:



$$\mathcal{H} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x, y \geq 0 \\ xy \geq 1 \end{matrix} \right\}$$

$$\mathcal{H} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \boxed{\begin{pmatrix} \boxed{x} & 1 \\ 1 & \boxed{y} \end{pmatrix}} \succcurlyeq 0 \right\}$$

determinants of all principal minors non-negative

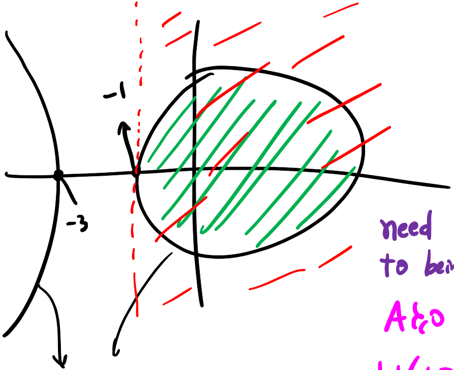
$x \geq 0 \quad y \geq 0$

$xy - 1 \geq 0$

Example of Spectrahedron

Elliptic curve: (oval part)

$$\mathcal{S} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; A = \begin{pmatrix} \boxed{x+1} & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{pmatrix} \succeq 0 \right\}$$



$$0 = \underbrace{-2y^2 - x^3 - 3x^2 + x + 3}_{\det(A)}$$

need to show that oval part corresponds to being in $\mathcal{S}$.

$A \succeq 0$ iff $\det(tI - A)$ has ≥ 0 roots

$$\det(tI-A) = t^3 - (x+5)t^2 + (-x^2+2x-y^2+7)t - \det(A)$$

$\Leftrightarrow$ $x+5 \geq 0$, $-x^2+2x-y^2+7 \geq 0$ and $\det(A) \geq 0$

$x+1 \geq 0 \Leftrightarrow x \geq -1$

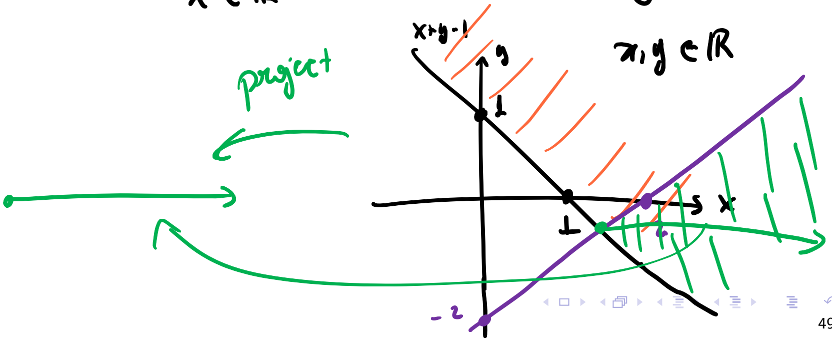
Projected Spectrahedron

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

$c^T\binom{x}{y}$ $\quad c = \binom{1}{0}$

$$\min\ c^T x \quad \text{s.t.}\ Ax \geq b,\ x \in \mathbb{R}^n$$

$$\min\ x \quad \text{s.t.}\ x+y \geq 1,\ x-y \leq 2,\ x,y \in \mathbb{R}$$



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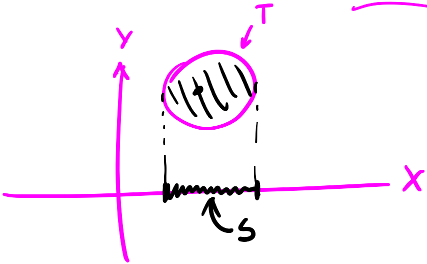
Definition (Projected Spectrahedron)

A set $S \in \mathbb{R}^n$ is a *projected spectrahedron* if it has the form:

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; \underbrace{\exists y \in \mathbb{R}^t}_{\text{auxiliary}} \text{ s.t. } \underbrace{\sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{t} B_j y_j \succeq C,}_{\text{LMI}} \quad A_i, B_j, C \in \mathcal{S}^m \right\}$$



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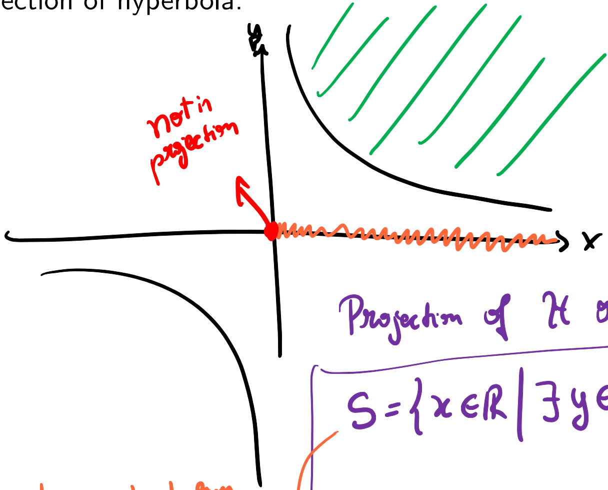
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & x \in S \end{aligned} \iff \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & (x, y) \in T \end{aligned}$$

$$T = \left\{ (x,y) \;\middle|\; \sum A_i x_i + \sum B_j x_j \succeq C \right\}$$

Example of Projected Spectrahedron

$\mathcal{H} = \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; \begin{pmatrix} x & 1\\ 1 & y\end{pmatrix} \succeq 0\right\}$

Projection of hyperbola:



Projection of $\mathcal{H}$ onto first coordinate (x)

$$S = \left\{x\in\mathbb{R} \;\middle|\; \exists\, y \in \mathbb{R} \text{ s.t. } \underbrace{\begin{pmatrix}x&1\\1&y\end{pmatrix}\succeq 0}_{\substack{x\geq 0 \quad y\geq 0\\ xy\geq 1}}\right\}$$

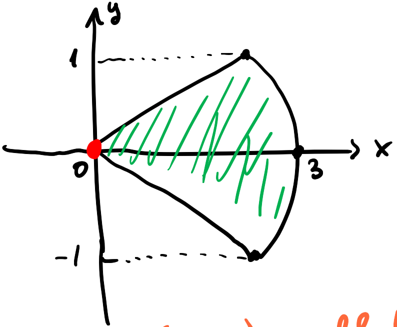
$$= \mathbb{R}_{>0}$$

not spectrahedron because it isn't closed

(spectrahedra are closed)

Example of Projected Spectrahedron

Projection quadratic cone intersected with halfspace:



$$S = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \exists z \in \mathbb{R} \text{ s.t. } \begin{pmatrix} z+y & 2z-x \\ 2z-x & z-y \end{pmatrix} \succcurlyeq 0 \;,\; z \leq 1 \right\}$$

in $\mathbb{R}^3$ (x,y,z) would be given by intersection of $z \leq 1$ with $z^2 \geq y^2 + (2z-x)^2$ and $\underbrace{z+y \geq 0 \quad z-y \geq 0}_{z \geq 0}$

these equations also imply that $x \geq 0$

This also not spectrahedron

How do we test membership in the Spectrahedron?

- To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

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- Symmetric Gaussian Elimination!
- We will use following characterizations of PSDness of symmetric $A \in \mathcal{S}^m$
  1. all eigenvalues of A are *non-negative*
  2. $A = LDL^T$ for some L lower triangular and unit diagonal, D diagonal and non-negative
  3. $z^T A z \geq 0$ for any $z \in \mathbb{R}^m$
  4. Any principal minor of A has non-negative determinant

How do we test membership in the Spectrahedron?

- **Input:** symmetric matrix $A \in \mathcal{S}^m$
- **Output:** YES if $A \succeq 0$, NO otherwise (and output $z \in \mathbb{R}^m$ such that $z^T A z < 0$)

↳ certifies that A is not PSD

Clear out first column of A by left multiplication

$$L_1 A = \begin{pmatrix} 1 & & & 0 \\ \alpha_2 & 1 & & \\ \alpha_3 & & \ddots & \\ \vdots & 0 & & \\ \alpha_m & & & 1 \end{pmatrix} \left(\begin{array}{c|c} A_{11} & A_1' \\ A_{21} & A_2' \\ \vdots & \vdots \\ A_{m1} & A_m' \end{array}\right) = \left(\begin{array}{c|c} A_{11} & A_1' \\ A_{21} - \alpha_2 A_{11} & A_2' - \alpha_2 A_1' \\ \vdots & \end{array}\right)$$

Call these values become 0

$$= \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & * & \cdots & & * \end{pmatrix}$$

A symmetric

$$L_1 A L_1^T = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}$$

symmetric dimension m-1.

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Repeat process of clearing out rows and columns

$$L_2 L_1 A L_1^T L_2^T = \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & \boxed{\begin{matrix} * & * & * \\ & \vdots & \\ * & * & * \end{matrix}} \end{pmatrix}$$

and so on

≥ 0

$$\underbrace{L_m L_{m-1} \cdots L_2 L_1}_{L \text{ unit lower triangular}} A \underbrace{L_1^T L_2^T \cdots L_m^T}_{L^T} = \begin{pmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix}$$

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our algorithm halts if the following happens:

if at any pt we have

$$
\begin{array}{cc}
LAL^T = \begin{pmatrix} * & & & & \\ & * & & 0 & \\ & & -a & 0 \cdots & 0 \\ & 0 & 0 & & \\ & & \vdots & \boxed{*} & \\ & & 0 & & \end{pmatrix} & \leftarrow i^{th} \text{ row} \\
\uparrow i^{th} \text{ column} &
\end{array}
$$

with $a > 0$ then return $\underline{NO}$

$z = L^T e_i$

$z^T A z = \underbrace{e_i^T L A L^T e_i}_{ii \text{ entry of } LAL^T} = -a < 0 \quad \therefore A$ is not PSD

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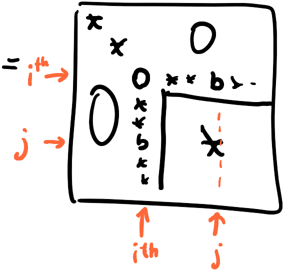
if at some pt $LAL^T =$



if $b \neq 0$ output NO

(determinant of principal minor $\{i,j\}$ is $-b^2 < 0$)

so LAL^T is not PSD

Practice problem: what is z in this case?

How do we test membership in the Spectrahedron?

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if our algorithm doesn't halt

then in the end we have

if >0

$$\begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & & \\ \vdots & & \vdots & & \\ 0 & \cdots 0 & 0 & 0 \cdots & 0 \\ & & \vdots & & \\ 0 & & 0 & & \end{pmatrix}$$

$LAL^T =$ diagonal and non-negative

$$A = (L^{-1}) D (L^{-1})^T$$

one of the equivalent PSD conditions

- Our algorithm runs in time strongly polynomial.

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- **Application: Control Theory**
- Conclusion
- Acknowledgements

Stability of Linear Systems

Setup:

- Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

- Discrete-time dynamical system.[1]

[1] When A non-negative and x_0 non-negative we have Markov chains.

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- System is stable iff $|\lambda_i(A)| < 1$

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SDP viewpoint:

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- To make these monotonically decreasing, we need:

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Theorem

Given matrix $A \in \mathbb{R}^{m \times m}$, the following conditions are equivalent:

1. *All eigenvalues of A are inside unit circle, i.e. $|\lambda_i(A)| < 1$*
2. *There is $P \in \mathcal{S}^m$ such that*

$$P \succ 0, \qquad A^T P A - P \prec 0$$

Where is the control?

Setup:

- Linear difference equation, with *control input*

$$x(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0$$

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- Want $P \succ 0$ such that

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- Wait, this ain't no SDP! But we can make it into SDP with some matrix manipulations.

Conclusion

- Mathematical programming - very general, and pervasive in Algorithmic life
- General mathematical programming very hard

[2] pun intended

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  - many more!
- Check out connections to Sum of Squares and a **bold**[2] attempt to have one algorithm to solve all problems! (i.e., one algorithm to rule them all)

https://windowsontheory.org/2016/08/27/proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/

[2] pun intended

Acknowledgement

- Lecture based largely on:
  - [Blekherman, Parrilo, Thomas 2012, Chapter 2]

References I

Blekherman, Grigoriy and Parrilo, Pablo and Thomas, Rekha (2012)
Convex Algebraic Geometry