Lecture 13: Linear Programming Relaxation and Rounding

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Overview

- Part I
  - Why Relax & Round?
- Vertex Cover
- Set Cover
- Conclusion
- Acknowledgements

Motivation - NP-hard problems

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- **Integer Linear Program (ILP):**

$$\text{minimize} \quad c^T x$$

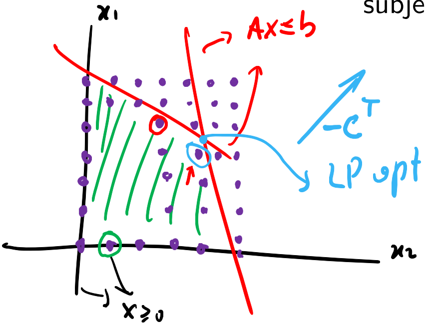
$$\text{subject to} \quad Ax \leq b$$

$$x \in \mathbb{N}^n$$

ILP

$x \geq 0 \quad x \in \mathbb{R}^n$

LP



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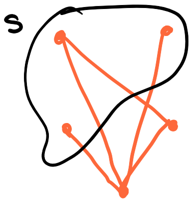
- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

Example



Maximum Independent Set:

$G(V, E)$ graph.

Independent set $S \subseteq V$ such that $u, v \in S \Rightarrow \{u, v\} \notin E$.

Integer Linear Program:

$$\text{maximize} \quad \sum_{v \in V} x_v$$

max |S|

$$\text{subject to} \quad x_u + x_v \leq 1 \quad \text{for } \{u, v\} \in E$$

$$x_v \in \{0, 1\} \quad \text{for } v \in V$$

← integral

for each edge $\{u,v\}$ of G at most one vertex in set S

$x_v = \begin{cases} 1 & \text{if we pick } v \in S \\ 0 & \text{o.w.} \end{cases}$

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2. Derive LP from the ILP by removing the integral constraints

This is called an *LP relaxation*.

Max independent set

$$\max \sum_{v \in V} x_v \iff \min -\sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \leq 1 \quad \forall \{u,v\} \in E$$

$$\cancel{x_u \in \{0,1\}} \iff \underbrace{0 \leq x_u \leq 1}_{\text{easier}}$$

LP relaxation

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3. We are still minimizing the same objective function, but over a (potentially) larger set of solutions.

$$opt(LP) \leq opt(ILP)$$

because solution space of LP is larger ("minimizing over bigger set")

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   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If solution has *fractional values*, then we have to devise *rounding procedure* that transforms

      fractional solutions → integral solutions

      feasible

$$opt(LP) \leq \text{rounded solution} \leq c \cdot opt(ILP)$$

feasible

approximation factor

Not all LPs created equal

When solving LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

it is important to understand *geometry of feasible set* & how nice the *corner points* are, as they are the candidates to *optimum* solution.

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- Let $P := \{x \in \mathbb{R}^n_{\geq 0} \mid Ax = b\}$

  feasible region (polytope)

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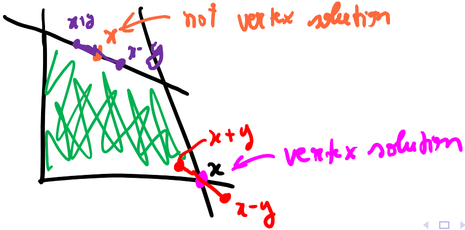
- Let $P := \{x \in \mathbb{R}^n_{\geq 0} \mid Ax = b\}$
- **Vertex Solutions:** a solution $x \in P$ is a vertex solution if $\nexists y \neq 0$ such that $x + y \in P$ *and* $x - y \in P$



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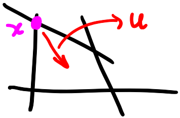

Not all LPs created equal

When solving LP

$x_1 \; x_2 \; x_3 \; x_4$

$x = (0, 1, \frac{1}{2}, 0)$

$supp(x) = \{2, 3\}$

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

describes P

all these definitions are equivalent

it is important to understand *geometry of feasible set* & how nice the *corner points* are, as they are the candidates to *optimum* solution.

- Let $P := \{x \in \mathbb{R}^n_{\geq 0} \mid Ax = b\}$
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- **Basic Solutions:** let $supp(x) := \{i \in [n] \mid x_i > 0\}$ be the set of nonzero coordinates of x. Then $x \in P$ is a basic solution $\Leftrightarrow$ the columns of A indexed by $supp(x)$ are linearly independent.

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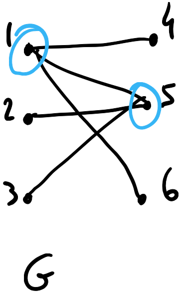
Vertex Cover

Setup:

- **Input:** a graph $G(V, E)$.
- **Output:** Minimum number of vertices that "touches" all edges of graph. That is, minimum set S such that for each edge $\{u, v\} \in E$ we have

$$|S \cap \{u, v\}| \geq 1.$$



$S_1 = \{1, 2, 3\}$ ✓ Covers

$S_2 = \{1, 5\}$

S_2 minimum cover

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1. Setup ILP:

$x_u = \begin{cases} 1 & \text{if add } u \in S \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{minimize} \quad & \sum_{u \in V} c_u \cdot x_u \\ \text{subject to} \quad & x_u + x_v \geq 1 \quad \text{for } \{u, v\} \in E \\ & x_u \in \{0, 1\} \quad \text{for } u \in V \end{aligned}$$

← total cost of set S

← inclusion in S

S has to cover every edge

Simple 2-approximation (unweighted)

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add both vertices to S

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Proof of correctness:

- By construction, S is a vertex cover.

if before seeing $\{u,v\}$ $S \cap \{u,v\} = \emptyset$
then after seeing $\{u,v\}$ we know
that $S \cap \{u,v\} \neq \emptyset$

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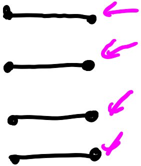
Proof of correctness:

- By construction, S is a vertex cover.
- If added elements to S k times, then $|S| = 2k$ and G has a matching of size k, which means that optimum vertex cover is at least k.

because every time we add to S we add 2 new vertices



Vertex cover must pick vertex from each edge of matching.

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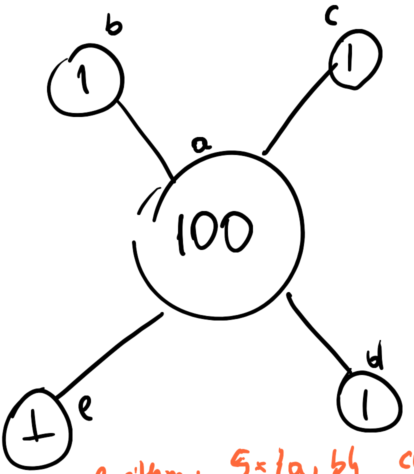
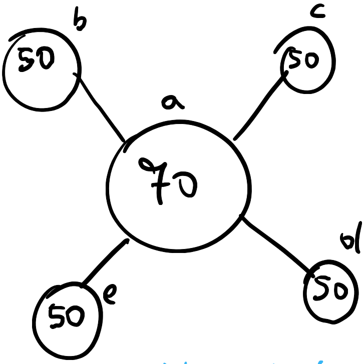
- Thus, we get a 2-approximation.

What can go wrong in the weighted case?

Original Algo



Heuristic: pick lowest weight only



$S = \{b, c, d, e\}$

$S^* = \{a\}$

our algorithm: $S = \{a, b\}$ $c(S) = 101$

optimum: $S^* = \{b, c, d, e\}$ $c(S^*) = 4$

Vertex Cover - LP relaxation

1. Setup ILP:

$$\begin{aligned} \text{minimize} \quad & \sum_{u \in V} c_u \cdot x_u \\ \text{subject to} \quad & x_u + x_v \geq 1 \quad \text{for } \{u, v\} \in E \\ & x_u \in \{0, 1\} \quad \text{for } u \in V \end{aligned}$$

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$z_u \in [0,1]$

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4. Round LP as follows: round z_v to nearest integer. $z_v \to \begin{cases} 1 & \text{if } z_v \geq 1/2 \\ 0 & \text{o.w.} \end{cases}$

Vertex Cover - Analysis

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3. Round z_v to nearest integer. That is $y_v = \begin{cases} 1, & \text{if } z_v \geq 1/2 \\ 0, & \text{if } 0 \leq z_v < 1/2 \end{cases}$

wait : is $\bar{y}$ a feasible solution ?

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5. each edge is covered, since given $\{u, v\} \in E$, at least one of z_u, z_v is $\geq 1/2$ (by feasibility of LP)

$z_u + z_v \geq 1 \Rightarrow$ one of $z_u, z_v \geq 1/2$

$\Rightarrow$ one of $y_u, y_v \geq 1$ $\therefore$ y_v cover

Vertex Cover - Analysis

$\sum_{u \in V} c_u y_u$

know: $y_u \le 2 z_u$

proof: $y_v = 1 \Leftrightarrow z_v \ge 1/2 \Leftrightarrow 2z_v \ge 1 = y_v$

$y_v = 0 \Leftrightarrow 0 \le z_v \le 1/2 \Rightarrow y_v = 0 \le 2z_v$

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4. y is an integral cover by construction
5. each edge is covered, since given $\{u, v\} \in E$, at least one of z_u, z_v is $\ge 1/2$ (by feasibility of LP)
6. Cost of y is:

$2 \sum c_u z_u$, $OPT(LP) \le OPT(ILP)$

$$\sum_{u \in V} c_u \cdot y_u \le \sum_{u \in V} c_u \cdot (2 \cdot z_u) \le 2 \cdot OPT(ILP)$$

$\ge y_u$

Set Cover

{1, 2, .., m}

"

$S_i \subset U$

Setup:

- **Input:** a finite set U and a collection $S_1, S_2, \ldots, S_n$ of subsets of U.
- **Output:** The fewest collection of sets $I \subseteq [n]$ such that

$$\bigcup_{j \in I} S_j = U.$$

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minimize total weight of collection

for each element $v \in U$ we must include a set which contains v

$$x_j = \begin{cases} 1 & \text{if } j \in I \\ 0 & \text{otherwise} \end{cases}$$

Set Cover - Relax...

1. Obtain LP relaxation:

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relax integral constraints

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2. Suppose we end up with fractional solution $z \in [0,1]^n$ when we solve the LP above. Now need to come up with a rounding scheme.

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2. Suppose we end up with fractional solution $z \in [0,1]^n$ when we solve the LP above. Now need to come up with a rounding scheme.
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Set Cover - Relax...

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4. Not really. Say $v \in U$ is in 20 sets, and we got $z_i = 1/20$ for each of the sets $v \in S_i$. Then rounding procedure above would not select any such set!

∴ rounded solution not feasible!

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4. *for* $i = 1, \ldots n$
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add set S_i to our collection

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5. *return I*

w.p. z_i add S_i
cost w_i

$\mathbb{E}[\text{cost}] = \sum_{i=1}^{n} z_i \cdot w_i$

4. Expected cost of the sets is $\sum_{i=1}^{n} w_i \cdot z_i$, which is the optimum for the LP. But will this process cover U?

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  Because z is a solution to our LP
- What is probability that v is covered in Random Pick?

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$v \in S_1, S_2$ $\Pr[\text{pick } S_i] = 1/2$

$\Pr[v \text{ not covered}] = \Pr[\text{didn't pick } S_1] \cdot \Pr[\text{didn't pick } S_2]$

$= (1 - z_1)(1 - z_2) = 1/4$

- Definitely not 1. Think about case $k = 2$ and $z_1 = z_2 = 1/2$.

$\Pr[v \text{ covered}] = 3/4$

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- By perseverance! :)

Probability that Element is Covered

Lemma (Probability of Covering an Element)

In a sequence of k independent experiments, in which the i^{th} experiment has success probability p_i, and

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then there is a probability $\geq 1 - 1/e$ that at least one experiment is successful.

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- $1 - x \leq e^{-x}$ for $x \in [0, 1]$
- Thus probability of failure is

$$\prod_{i=1}^{k} (1 - p_i) \leq \prod_{i=1}^{k} e^{-p_i} = e^{-p_1 - \cdots - p_k} \leq 1/e$$

≤ -1

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   } do an iteration of random pick
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random pick

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Let $t \in \mathbb{N}$. *The probability that the for loop will be executed more than* $\ln(|U|) + t$ *times is at most* e^{-t}.

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- Probability that v not covered after $\ln(|U|) + t$ iterations is independent

$$\left(\frac{1}{e}\right)^{\ln(|U|)+t} = \frac{1}{|U|} \cdot e^{-t}$$

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- Union bound. $\leq |U| \cdot \frac{1}{|U|} \cdot e^{-t} = e^{-t}$

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$\leq 0.05 + 0.5 = 0.55$

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3. Markov $\Rightarrow$ probability that our solution has weight $\geq 2 \cdot \omega$ is $\leq 1/2$
4. Union bound, with probability ≤ 0.55 either run for more than t times, or our solution has weight $\geq 2\omega$
5. Thus, with probability ≥ 0.45 we stop at t iterations **and** construct solution to set cover with cost $\leq 2t \cdot OPT(ILP)$

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$$OPT(LP) \leq OPT(ILP)$$

$\sum w_i z_i$

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    1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
    2. If have *fractional values*, *rounding procedure*

    Randomized Rounding algorithm, with probability ≥ 0.45 we get

    constant

$$cost(\text{rounded solution}) \leq 2 \cdot (\ln(|U|) + 3) \cdot OPT(ILP)$$

approximation factor

Conclusion

- Integer Linear programming - very general, and pervasive in (combinatorial) algorithmic life
- ILP NP-hard
- Rounding for the rescue!
- Solve LP and round the solution
    - Deterministic rounding when solutions are nice HW problem 1
    - Randomized rounding when things a bit more complicated

core for vertex cover

set cover

Acknowledgement

- Lecture based largely on:
  - Lectures 7-8 of Luca's Optimization class
- See Luca's vertex cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture07.pdf
- See Luca's set cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture08.pdf