Lecture ~~12~~ 11: Linear Programming and Duality Theorems

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Overview

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Conclusion
- Acknowledgements
- Proof of Fundamental Theorem of Linear Inequalities

Mathematical Programming

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- Very general family of problems.
- Special case is when all functions $f, g_1, \ldots, g_m$ are *linear* functions (called *Linear Programming* - LP for short)
- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n = c^T x \quad + b$$

$+ b$

$b \in \mathbb{R}$

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$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & A_1^T x \leq 0 \;\; b_1 \\ & \vdots \\ & A_m^T x \leq 0 \;\; b_m \\ & x \in \mathbb{R}^n \end{aligned}$$

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \end{pmatrix} \qquad A^T \vec{x} \leq \vec{b}$$

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We can *always* represent LPs in *standard form*:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \ge 0 \end{aligned}$$

$a_{i1}x_1 + \cdots + a_{in}x_n + s_i = b_i$

$s_i \ge 0$

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!

- Linear Programming is also a great theoretical tool to prove some really cool results!

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  - n companies, stock of company i costs $c_i \in \mathbb{R}$
  - company i has expected profit $p_i \in \mathbb{R}$
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expected profit

$$\begin{aligned} \text{maximize} \quad & p_1 \cdot x_1 + \cdots + p_n \cdot x_n \\ \text{subject to} \quad & c_1 \cdot x_1 + \cdots + c_n \cdot x_n \leq B \\ & x \geq 0 \end{aligned}$$

amount of shares that we have must fit our budget

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!
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    - n companies, stock of company i costs $c_i \in \mathbb{R}$
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- Other problems, such as *data fitting, linear classification* can be modelled as linear programs.

Important Questions

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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1. When is a Linear Program *feasible*?
   - Is there a solution to the constraints at all?

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3. Can we characterize *optimality*?
   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?

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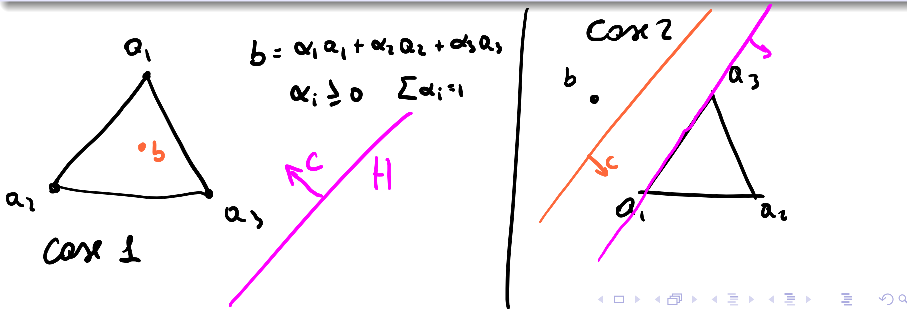
    - Do these solutions have nice description?
    - Do the solutions have *small bit complexity*?
4. How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs?

Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, *and* $t := \text{rank}\{a_1, \ldots, a_m, b\}$. *Then either*

1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or*
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
   - *H contains $t - 1$ linearly independent vectors from $a_1, \ldots, a_m$*



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Remark

The hyperplane H above is known as the *separating hyperplane*.

Farkas' Lemma

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *$y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$*

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Equivalent formulation

Lemma (Farkas Lemma - variant 1)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$*

Farkas' Lemma

Equivalent formulation

Lemma (Farkas Lemma - variant 2)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $Ax \leq b$*
2. *$y^T b \geq 0$ for each $y \geq 0$ such that $y^T A = 0$*

$$Ax \leq b \underset{y \geq 0}{\Rightarrow} \underbrace{y^T A}_{0} x \leq y^T b$$

$$\Downarrow$$

$$0 \leq y^T b$$

Farkas' Lemma

$$s_i + [A(p-n)]_i = b_i$$

$$A(p-n) \leq b \quad \leftrightarrow \quad Ax \leq b$$

$$x = p - n$$

Equivalent formulation

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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$

$$M = \begin{pmatrix} I & A & -A \end{pmatrix} \qquad Mz = b$$

$$z = \begin{pmatrix} s \\ p \\ n \end{pmatrix} \qquad \boxed{Is + A\cdot p - An = b}$$

Farkas' Lemma

Equivalent formulation

Lemma (Farkas Lemma - variant 2)

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1. *There exists $x \in \mathbb{R}^n$ such that $Ax \leq b$*
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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$
- Now apply the original version of the lemma

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Linear Programming Duality

Consider our linear program:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\boxed{y^T A \leq c^T} \Rightarrow y^T A x \leq c^T x$$
$$\Rightarrow y^T b \leq c^T x$$

because $x \geq 0$

standard form $Ax = b$

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- If we look at what happens when we multiply $y^T A$, note the following:

holds for any solution to our LP

$$\begin{aligned} y^T A \leq c^T &\Rightarrow y^T A x \leq c^T x \\ &\Rightarrow y^T b \leq c^T x \end{aligned}$$

↳ objective function

- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

Linear Programming Duality

Consider the following linear programs:

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

any y satisfying the constraint

$\Rightarrow$ $y^T b$ lower bd on primal

dual LP is maximizing lower bound to the primal via $y^T A \leq c^T$

Linear Programming Duality

Consider the following linear programs:

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
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- From previous slide

$$y^T A \leq c^T \Rightarrow y^T b \text{ is a lower bound on value of Primal}$$

Linear Programming Duality

Consider the following linear programs:

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- Thus, the optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!

Weak duality of LP

Linear Programming Duality

Consider the following linear programs:

Primal LP	*Dual LP*
minimize $c^T x$	maximize $y^T b$
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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

Remarks on Duality

Primal LP	solutions	*Dual LP*
minimize $c^T x$	$\geq$	maximize $y^T b$
subject to $Ax = b$		subject to $y^T A \leq c^T$
$x \geq 0$		

- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!

Remarks on Duality

Primal LP			*Dual LP*		
α^*	minimize	$c^T x$	β^*	maximize	$y^T b$
	subject to	$Ax = b$		subject to	$y^T A \leq c^T$
		$x \geq 0$			

- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!
- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.

Remarks on Duality

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Dual LP

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- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.
  - We showed that when both primal and dual are feasible, we have

$$\text{max dual} \ = \beta^* \leq \alpha^* = \ \text{min of primal}$$

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  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)

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  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)

Remarks on Duality

Primal LP		*Dual LP*	
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  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)
- **Practice problem:** show that dual of the dual LP is the primal LP!
- When is the above inequality tight?

Strong Duality

Primal LP	
minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

Dual LP	
maximize	$y^T b$
subject to	$y^T A \leq c^T$

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Strong Duality

Primal LP

minimize $c^T x$

subject to $Ax = b$

$x \geq 0$

Dual LP

maximize $y^T b$

subject to $y^T A \leq c^T$

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Theorem (Strong Duality)

If primal LP and dual LP are feasible, then

$$\text{max dual } = \beta^* = \alpha^* = \text{ min of primal.}$$

i.e.: both programs have the same value!

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP and dual LP are feasible, then

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Proof of Strong Duality

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If primal LP and dual LP are feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal.}$$

1. Since we have proved weak duality, suffices to show that the following LP has a solution:

↪ x, y feasible then $y^T b \le c^T x$

don't care about what we are maximizing ⟷

$$
\begin{aligned}
\text{maximize} \quad & 0 \\
\text{subject to} \quad & y^T A \le c^T \\
& c^T x - y^T b \le 0 \\
& Ax = b \\
& x \ge 0
\end{aligned}
$$

dual — y sol. to the dual

primal — x sol. to primal

Primal	Dual
min $c^T x$	max $y^T b$
s.t. $Ax = b$, $x \ge 0$	s.t. $y^T A \le c^T$

$c^T x \le y^T b$ ⟺ x, y are optimum solutions

Proof of Strong Duality

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If primal LP and dual LP are feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal.}$$

1. Since we have proved weak duality, suffices to show that the following LP has a solution:

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & y^T A \leq c^T \\ & c^T x - y^T b \leq 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

2. Apply variant 2 of Farkas' lemma on the system above.

Proof of Strong Duality

1. LP from previous page encoded by:

$$B \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & A^T \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ 0 \\ c \end{pmatrix}$$

$$\left.\begin{array}{l} Ax \leq b \\ -Ax \leq -b \iff Ax \geq b \end{array}\right\} Ax = b$$

$$c^T x - b^T y \leq 0 \iff c^T x \leq y^T b$$

$$A^T y \leq c \iff y^T A \leq c^T$$

$$\begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & -A^T \\ -I & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ 0 \\ c \\ 0 \end{pmatrix}$$

$$z = \begin{pmatrix} u^T & v^T & \lambda & w^T & \alpha^T \end{pmatrix} \geq 0$$

$$zB = (0, 0) \Rightarrow \underbrace{u^T b - v^T b + w^T c \geq 0}_{\alpha,\ \lambda \text{ do not appear}}$$

$$zB = (0, 0) \quad u^T A - v^T A + \lambda c^T - \alpha^T = 0 \Leftrightarrow \boxed{u^T A - v^T A + \lambda c^T \geq 0}$$

Proof of Strong Duality

1. LP from previous page encoded by:

$$B\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}A & 0\\ -A & 0\\ c^T & -b^T\\ 0 & A^T\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} \leq \begin{pmatrix}b\\-b\\0\\c\end{pmatrix}$$

2. Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \;\; v^T \;\; \lambda \;\; w^T) \geq 0$ such that $zB = 0$ then we have $u^Tb - v^Tb + w^Tc \geq 0$

$$\left.\begin{array}{l} zB = 0\\ z \geq 0\end{array}\right\} \Rightarrow z\begin{pmatrix}b\\-b\\0\\c\end{pmatrix} \geq 0$$

(variant 2 of Farkas lemma)

$$(u^T \;\; v^T \;\; \lambda \;\; w^T)\begin{pmatrix}b\\-b\\0\\c\end{pmatrix} = u^Tb - v^Tb + w^Tc \geq 0$$

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3. If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so

$$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$

$$= (u^T - v^T)[\lambda b - Aw] = 0$$

$$\lambda c^T w = (v^T - u^T)Aw$$

$$\lambda > 0 \Rightarrow (u^T - v^T)b + w^T c \geq 0 \Leftrightarrow \lambda[(u^T - v^T)b + w^T c] \geq 0$$

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4. If $\lambda = 0$, let x, y be feasible solutions (which we assumed to exist). Then $x \geq 0, Ax = b$ and $y^T A \leq c^T$. Thus

$$c^T w \geq y^T Aw = 0 = (v^T - u^T)Ax = (v^T - u^T)b$$

$w \geq 0$

Proof Strong Duality: $\lambda > 0$

$$zB = \begin{pmatrix} u^T & v^T & \lambda & w^T \end{pmatrix} \begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & A^T \end{pmatrix} =$$

$$\| $$

$$(0, 0)$$

$$= \left((u^T - v^T)A + \lambda c^T \,,\; w^T A^T - \lambda b^T \right)$$

$$\Longleftrightarrow \quad \lambda c^T \geq (v^T - u^T)A \quad \text{and} \quad \lambda b^T = w^T A^T$$

$$\Updownarrow$$

$$\boxed{\lambda b = A w}$$

Proof of Strong Duality: $\lambda = 0$

$\lambda = 0$ and $zB = 0$ $\Longleftrightarrow$ $(v^T - u^T)A \leq 0$ and $w^T A^T = 0$ $\Leftrightarrow$ $Aw = 0$

$$c^T w \geq \underbrace{y^T A w}_{0} = 0 \geq \underbrace{(v^T - u^T)A}_{\leq 0} \underbrace{x}_{\geq 0} = (v^T - u^T) b$$

$\uparrow$ $y^T A \leq c$, $w \geq 0$

$\uparrow$ $Ax = b$ (x feasible solution of primal)

rearranging $c^T w + (u^T - v^T) b \geq 0$ ☒

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \leq b$$

have at least one solution, and suppose that inequality

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holds whenever x *satisfies* $Ax \leq b$. *Then, for some* $\delta' \leq \delta$ *the linear inequality*

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Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Complementary Slackness

- If the optima in both primal and dual is finite, and x, y are feasible solutions, the following are equivalent:
  1. x, y are optimal solutions to the primal and dual
  2. $c^T x = y^T b$
  3. if $x_i > 0$ then the corresponding inequality $y^T A_i \leq c_i$ is an equality: that is, we must have $y^T A_i = c_i$.

strong duality

(3) for every $x_i > 0 \Rightarrow y^T A_i = C_i$

3 equivalent to saying x, y are both optima

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- 1 and 2 are equivalent due to strong duality
- 2 and 3 are equivalent as we can write

$$c^T x - y^T b = c^T x - y^T A x = (c^T - y^T A)x = \sum_{i=1}^{n} (c_i - y^T A_i) x_i$$

↑ x feasible ∴ $Ax = b$ $=0$ >0

$x_i \geq 0$ condition 3 $\Leftrightarrow \sum_{i=1}^{n} x_i (c_i - y^T A_i) = 0 \Leftrightarrow c^T x - y^T b = 0$

Conclusion

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- General mathematical programming very hard (how hard do you think it is?)
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Today: *Linear Programming*

- Linear Programming and Duality - fundamental concepts, lots of applications!
  - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
  - Applications in Game Theory (minimax theorem)
  - Applications in Learning Theory (boosting)
  - many more

Acknowledgement

- Lecture based largely on:
    - [Schrijver 1986, Chapter 7]

Proof of Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, *and* $t := \text{rank}\{a_1, \ldots, a_m, b\}$. *Then either*

1. b *is a non-negative linear combination of linearly independent vectors from* $a_1, \ldots, a_m$, *or*
2. *there exists a hyperplane* $H := \{x \mid c^T x = 0\}$ *s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
   - H *contains* $t - 1$ *linearly independent vectors from* $a_1, \ldots, a_m$

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- Since 1 and 2 mutually exclusive, choose linearly independent $\mathcal{L}_0 := \{a_{i_1}, \ldots, a_{i_n}\}$
- We will perform an iterative procedure:

Proof of Fundamental Theorem of Linear Inequalities

Iterative procedure, starting with $\mathcal{L}_0$:

1. Write $b = \lambda_{i_1} a_{i_1} + \ldots + \lambda_{i_n} a_{i_n}$. If $\lambda_i \geq 0$ we are done

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3. If $c_0^T a_i \geq 0$ for all $i \in [m]$ we are done (case 2)

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- To conclude the proof, need to show that this procedure always terminates. If process doesn't terminate, there are two times $r < t$ such that $\mathcal{L}_r = \mathcal{L}_t$
- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.

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- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.
- $\mathcal{L}_r = \mathcal{L}_t \Rightarrow a_\ell$ has also been added from some $\mathcal{L}_{k'}$ for some $r \leq k' < t$.

Proof of Fundamental Theorem of Linear Inequalities

- Say a_r was removed at iteration k and added back at iteration k' so $r \leq k < k' < t$
- Let c be the vector defining the hyperplane at the k' iteration (when we added a_r back to the set), and let $\mathcal{L}_k = \{a_{i_1}, \ldots, a_{i_n}\}$
- Now, above implies the following contradiction:

$$0 > c^T b = c^T(\lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} c^T a_{i_1} + \cdots \lambda_{i_n} c^T a_{i_n} \geq 0$$

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- Second inequality holds term by term:
  -

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