Lecture 10: Fundamental Theorem of Markov Chains, Page Rank

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Overview

- Main Tools
  - Linear Algebra Background
  - Perron-Frobenius
- Main Applications
  - Fundamental Theorem of Markov Chains
  - Page Rank
- Conclusion
- Acknowledgements

Eigenvalues, Eigenvectors and Spectral Radius

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there is a vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

v is an eigenvector corresponding to λ

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- The *spectral radius* of a matrix A, denoted $\rho(A)$, is the maximum absolute value of the eigenvalues of A

$$\rho(A) = \max_{\lambda \text{ eigenvalue of } A} |\lambda|$$

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- *Gelfand's formula*

$$\rho(A) = \lim_{n \to \infty} \|A^n\|_F^{1/n}$$

$$\|A\|_F = \mathrm{tr}[A^T A]^{1/2}$$

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- *Geometric multiplicity*: an eigenvalue λ of A has geometric multiplicity k if the space of eigenvectors of A with eigenvalue λ has dimension k. That is, if dimension of null space of $A - \lambda I$ is k.

$$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix}$$

geometric multiplicity of 1 is 2

$\alpha e_1 + \beta e_2$ is eigenvector of 1

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- *Algebraic multiplicity*: an eigenvalue λ of A has algebraic multiplicity k if $(t-\lambda)^k$ is the highest power of $t-\lambda$ dividing $\det(tI - A)$ $= p(t)$

characteristic polynomial of A

$A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 3 \end{pmatrix}$ $\det(tI - A) = (t-1)^2(t-3)$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\det(tI - A) = (t-1)^2 \nearrow \text{algebraic mult. of } 1 \text{ is } 2$$

A only has 1 eigenvector : e_1 (eigenvalue is 1)

$\therefore$ geometric multiplicity is 1.

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

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- Example:

Positivity Lemma

Lemma (Positivity Lemma)

If $A \in \mathbb{R}^{n \times n}$ is a positive matrix and $u, v \in \mathbb{R}^n$ are distinct vectors such that $u \geq v$, then $Au > Av$. Moreover, there exists $\varepsilon > 0$ such that $Au > (1 + \varepsilon)Av$.

A is positive matrix if every entry

$A_{ij} > 0$.

$u \geq v \iff$ each coordinate $u_i \geq v_i$

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- Note that

$$A(u - v) = \sum_{j} A_{ij}(u_j - v_j) \geq (\min_{i,j} A_{ij}) \cdot \sum_{j} (u_j - v_j)$$

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$$\to \left[A(u-v)\right]_i = \sum_j A_{ij}(u_j - v_j) \geq (\min_{i,j} A_{ij}) \cdot \sum_j (u_j - v_j)$$

$> \min A_{ij} \cdot \mu > 0$

- Since $u_j \geq v_j$ for all j and u, v distinct implies that there is one index k such that $u_k > v_k$, we have

$$\sum_j (u_j - v_j) \geq u_k - v_k > 0$$

$$\left[A(u-v)\right]_i > \mu \cdot \min_{i,j} A_{ij} > 0$$

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- Note that

$$A(u-v) = \sum_{i,j} A_{ij}(u_j - v_j) \geq n \cdot (\min_{i,j} A_{ij}) \cdot \sum_j (u_j - v_j)$$

- Since $u_j \geq v_j$ for all j and u, v distinct implies that there is one index k such that $u_k > v_k$, we have

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- the moreover part just follows from taking small enough ε.

Perron's Theorem

Theorem (Perron's Theorem)

Let $A \in \mathbb{R}^{n \times n}$ be a positive matrix (i.e., all its entries are positive). Then, the following hold:

1. $\rho(A)$ *is an eigenvalue, and it has a positive eigenvector*
2. $\rho(A)$ *is the only eigenvalue in the complex circle* $|\lambda| = \rho(A)$
3. $\rho(A)$ *has geometric multiplicity 1*
4. $\rho(A)$ *has algebraic multiplicity 1*

(2) says that $\rho(A)$ is <u>only</u> eigenvalue of <u>max</u> absolute value

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- By the definition of $\rho(A)$, there is an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| = \rho(A)$. Let v the a corresponding eigenvector.

$$\rho(A) = \max_{\lambda \text{ eigenvalue of } A} |\lambda|$$

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- By the definition of $\rho(A)$, there is an eigenvalue $\lambda \in \mathbb{C}$ such that $|\lambda| = \rho(A)$. Let v the a corresponding eigenvector.
- Let u be the vector defined by $u_i = |v_i|$. Then, we have

triangle inequality

$$(Au)_i = \sum_j A_{ij} u_j \geq \left| \sum_j A_{ij} v_j \right| = |\lambda v_i| = \rho(A) \cdot u_i$$

$\to (Av)_i$

so $Au \geq \rho(A) u$.

$\langle A_i, u \rangle$ $\quad \sum A_{ij} \cdot |v_j| = \sum |A_{ij} v_j|$

Perron's Theorem - item 1

- We proved $Au \geq \rho(A)u$.
- If inequality strict, then we have

$$A^2u > \rho(A) \cdot Au$$

and there is some positive $\varepsilon > 0$ such that

$$A^2u \geq (1+\varepsilon)\rho(A)Au$$

$Au > \rho(A)u$

$A \cdot (Au)$

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$A \cdot (Au) \geq A \cdot (\rho(A)u) \cdot (1+\epsilon)$

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- By induction, we would have

$$A^{n+1} u \geq (1+\varepsilon)^n \cdot \rho(A)^n \cdot Au$$

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- By induction, we would have

$$A^n u \geq (1 + \varepsilon)^{n-1} \cdot \rho(A)^{n-1} \cdot Au$$

- By Gelfand's formula we would have

$$\rho(A) = \lim_{n \to \infty} \|A^n\|_F^{1/n} \geq (1 + \varepsilon)\rho(A)$$

$$\geq [(1+\varepsilon)^{n-1} \cdot \rho(A)^{n-1}]^{1/n} \quad \therefore Au = \rho(A)u$$

which is a contradiction. So equality must hold.

Perron's theorem - items 1 and 2

- We just proved that $\rho(A)$ is an eigenvalue, with eigenvector $u \geq 0$.

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- Note that $u > 0$ since $\rho(A)u_i = (Au)_i > 0$

$$Au = \rho(A)u \Rightarrow (Au)_i = \sum_{j=1}^{n} A_{ij} u_j \geq \underbrace{A_{ik}}_{>0} \underbrace{u_k}_{>0} > 0$$

$$(Au)_i = \rho(A) \cdot u_i$$

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- If we had another eigenvalue $\lambda \neq \rho(A)$ in the circle $|\mu| = \rho(A)$, where z is the eigenvalue (vector) corresponding to λ, by the previous slide, we know that w defined as $w_i = |z_i|$ satisfies

$Az = \lambda z$

$$Aw = \rho(A)w \quad \Leftrightarrow \quad \sum_j A_{ij} w_j = \rho(A) \cdot |z_i| = |\lambda z_i| = |\sum_j A_{ij} z_j|$$

$(Aw)_i = \rho(A) \cdot w_i$ $\quad$ $|\lambda| = \rho(A)$ $\quad$ λ eigenvalue, z eigenvector of λ

for every $1 \leq i \leq n$

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- *Lemma*: if the conditions above hold, then there is $\alpha \in \mathbb{C}$ nonzero such that $\alpha z \geq 0$

Proof by squaring both sides and using complex conjugates.

Perron's theorem - items 2 and 3

- But if $\alpha z \geq 0$ and a nonzero vector, we have

$$\lambda(\alpha z) = \alpha \cdot (\lambda z) = \alpha(Az) = A(\alpha z) \geq 0$$

≥ 0

positive

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- Thus we know that λ is a non-negative number. However, $\rho(A)$ is the only non-negative number in the circle $|\mu| = \rho(A)$. This concludes item 2.

$$\underbrace{\lambda}_{\geq 0} \cdot \underbrace{(\alpha z_1)}_{> 0} \geq 0$$

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- Now we are ready to prove item 3: the geometric multiplicity of $\rho(A)$ is 1.
- Suppose not, and let u, v be two linearly independent eigenvectors for $\rho(A)$. We can assume that both u, v are real vectors (why?).

$$A(\underline{\phi} + i\underline{\psi}) = \rho(A) \cdot \underline{\phi} + i\rho(A)\underline{\psi}$$

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- Suppose not, and let u, v be two linearly independent eigenvectors for $\rho(A)$. We can assume that both u, v are real vectors (why?).
- Let $\beta > 0$ be such that $u - \beta v \geq 0$ and at least one entry is zero.
- $u - \beta v \neq 0$ since the vectors are linearly independent
- But for each $1 \leq i \leq n$

$$\rho(A) \cdot (u - \beta v)_i = (A(u - \beta v))_i > 0$$

which contradicts our choice of β. Thus, there cannot be two linearly independent eigenvectors.

Perron-Frobenius

Theorem (Perron-Frbenius)

If a non-negative matrix $A \in \mathbb{R}^{n \times n}$ is aperiodic and irreducible, then the following hold:

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- By previous lecture, we saw that A being aperiodic and irreducible implies that there is $m > 0$ such that A^m has all positive entries.
- Apply Perron's theorem to A^m and note that the eigenvalues of A^m are λ_i^m, where λ_i are the eigenvalues of A

Fundamental Theorem of Markov Chains

- The *return time* from state i to itself is defined as

$$H_{i,i} := \min\{t \geq 1 \mid X_t = i, X_0 = i\}$$

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Theorem (Fundamental Theorem of Markov Chains)

Any finite, irreducible and aperiodic Markov Chain has the following properties:

1. *There exists a unique stationary distribution π, where $\pi_i > 0$ for all $i \in [n]$*
2. *The sequence of distributions $\{p_t\}_{t \geq 0}$ will converge to π, no matter what the initial distribution is*
3. $$\pi_i = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{h_{i,i}}$$

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- The transition matrix P is non-negative, irreducible and aperiodic. So we can apply Perron-Frobenius and prove items 1 and 2.

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If our underlying graph is undirected:

- If A_G adjacency matrix of $G(V,E)$ and $D = diag(d_1, d_2, \ldots, d_n)$, transition matrix:

$$P = D^{-1} \cdot A_G$$

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- Note that in this case, easy to guess stationary distribution:

$$\pi_i = \frac{d_i}{2m}, \quad m = |E|$$

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  - P not symmetric, but *similar* to a symmetric matrix:

$$D^{1/2} P D^{-1/2} = D^{1/2} D^{-1} A_G D^{-1/2} = D^{-1/2} A_G D^{-1/2} = P'$$

symmetric

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  - P and P' has *same eigenvalues*! And P' has only *real eigenvalues*!

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  - Eigenvectors of P are $D^{-1/2} v_i$ where v_i are eigenvectors of P'. And v_i can be taken to form *orthonormal basis*.

Fundamental Theorem of Markov Chains

- Stationary distribution: $\pi_i = \dfrac{d_i}{2m}, \quad m = |E|$
- Transition matrix: $P = D^{-1} \cdot A_G$
  - P *similar* to a symmetric matrix: $P' = D^{-1/2} A_G D^{-1/2}$
  - P and P' has *same eigenvalues*! And P' has only *real eigenvalues*!
  - Eigenvectors of P are $D^{-1/2} v_i$ where v_i are eigenvectors of P'. And v_i can be taken to form *orthonormal basis*.

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  - All random walks converge to π, as we wanted to show.

- Main Tools
  - Linear Algebra Background
  - Perron-Frobenius
- Main Applications
  - Fundamental Theorem of Markov Chains
  - Page Rank
- Conclusion
- Acknowledgements

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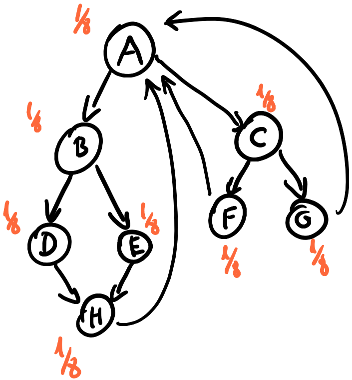
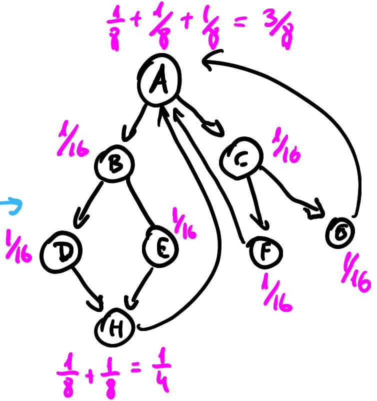
Page Rank - Example





$w_0 \cdot P = w_1$

↳ depends only on graph structure

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Page rank is a Markov chain!

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- If graph finite, irreducible and aperiodic, fundamental theorem guarantees stationary distribnution.

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- This modification does not change "relative importance" of vertices

Conclusion

Markov Chains and Random Walks are ubiquitous in randomized algorithms.

- Page Rank algorithm (today's lecture)
- Approximation algorithms for counting problems [Karp, Luby & Madras]
  - Permanent of non-negative matrices [Jerrum, Vigoda & Sinclair]
- Sampling Problems
  - Gibbs sampling in statistical physics
  - many more places
- Probability amplification without too much randomness (efficient)
  - Random walks on expander graphs
- many more

Acknowledgement

- Lecture based largely on:
  - Hannah Cairns notes on Perron-Frobenius (see link in course webpage)
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 6]
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf`
- Also see Lap Chi's notes `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf` for a proof of fundamental theorem of Markov chains for undirected graphs.

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