Lecture 9: Random Walks, Markov Chains, Mixing Time

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Overview

- Introduction
  - Why Random Walks & Markov Chains?
  - Basics on Theory of Finite Markov Chains
- Main Topics
  - Stationary Distributions and Mixing Time
  - Fundamental Theorem of Markov Chains
- Conclusion
- Acknowledgements

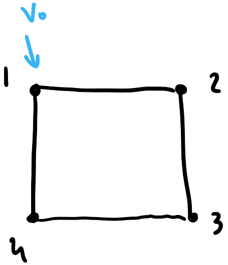
What is a Random Walk?

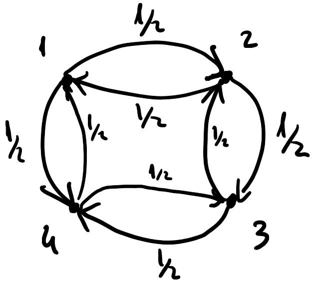
vertices of graph ↔ state of random walk

Given a graph $G(V, E)$

1. random walk starts from a vertex v_0
2. at each time step it moves *uniformly* to a *random neighbor* of the current vertex in the graph

$$v_{t+1} \leftarrow_R N_G(v_t)$$





1 → 2 → 1 → 2 → 3 → 4

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Basic questions involving random walks:

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Basic questions involving random walks:

- *Stationary distribution:* does the random walk converge to a "stable" distribution? If it does, what is this distribution?

if random walk has been happening for a while
what's is the probability of being in a given vertex?

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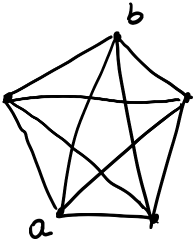
- *Hitting time:* starting from a vertex v_0, what is expected number of steps until it reaches a vertex v_f?
- *Cover time:* how long does it take to reach every vertex of the graph at least once?

Random Walk: Example

- Suppose $G(V, E) = K_n$, the complete graph, $a, b \in V$ two vertices



K_5

Random Walk: Example

- Suppose $G(V,E) = K_n$, the complete graph, $a, b \in V$ two vertices
    1. What is expected number of steps to reach b in simple random walk starting at a? (i.e., hitting time)

$$a \neq b$$

$$\Pr[a \to b] = \frac{1}{n-1}$$

$$E[T_{a,b}] = \sum_{k \geq 1} k \cdot \underbrace{\Pr[T_{a,b} = k]}_{\left(\frac{n-2}{n-1}\right)^{k-1} \cdot \frac{1}{n-1}} = \sum_{k \geq 1} k \cdot \left(\frac{n-2}{n-1}\right)^{k-1} \cdot \frac{1}{n-1}$$

$$= \sum_{k \geq 1} \underbrace{\Pr[T_{a,b} \geq k]}_{= \sum_{j \geq n} \Pr[T_{a,b} = j]} = \sum_{n \geq 1} \left(\frac{n-2}{n-1}\right)^{k-1} = \frac{1}{1 - \frac{n-2}{n-1}}$$

$$= n-1$$

Random Walk: Example

- Suppose $G(V, E) = K_n$, the complete graph, $a, b \in V$ two vertices
  1. What is expected number of steps to reach b in simple random walk starting at a? (i.e., hitting time)
  2. Starting from a, what is expected number of steps to visit all vertices? (i.e, cover time)

Practice problem: (you have seen this before)

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  3. Stationary Distribution?

Yes: $\boxed{\pi = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)}$ uniform distribution

$$\Pr\left[\text{being at } a \text{ at time } t \;\middle|\; \text{uniform at time } t-1\right] =$$

$$= \sum_{i=1}^{n} \Pr\left[X_t = a \mid X_{t-1} = i\right] \underbrace{\Pr\left[X_{t-1} = i\right]}_{1/n} = 0 + \underbrace{\frac{1}{n} \cdot \sum_{i \neq a} \frac{1}{n-1}}_{=1}$$

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- **Practice question:** Compare question 2 to coupon collector problem!

What is a Markov Chain?

Random walk is a special kind of *stochastic process*:

$$\Pr[X_t = v_t \mid \underbrace{X_0 = v_0, \ldots, X_{t-1} = v_{t-1}}_{\text{history}}] = \Pr[X_t = v_t \mid \underbrace{X_{t-1} = v_{t-1}}_{\text{random walk only cares where we were last}}]$$

$X_t \leftarrow \underbrace{\text{state}}_{\text{vertex in graph}}$ of random walk at time t

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Process is "*forgetful*/*memoryless*"

Markov chain is characterized by this property.

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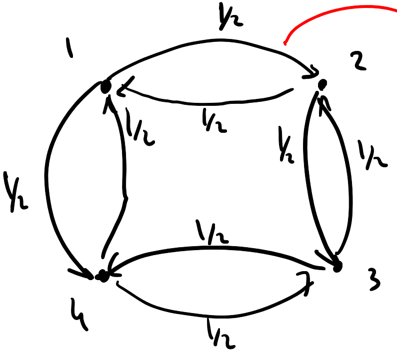
Representing Finite Markov Chains

Markov chain can be seen as weighted directed graph.

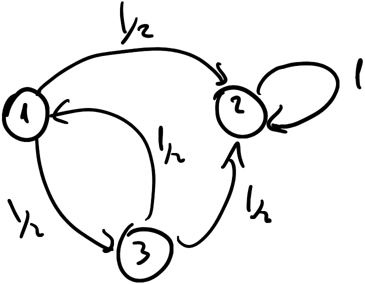


$\Pr[1 \rightsquigarrow 2] = 1/2$

Representing Finite Markov Chains

Markov chain can be seen as weighted directed graph.

- Vertex is a state of Markov chain
- edge (i, j) corresponds to transition probability from i to j



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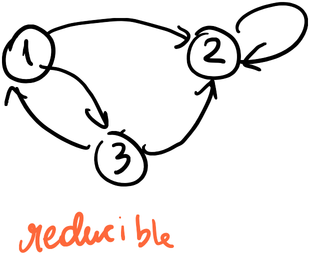

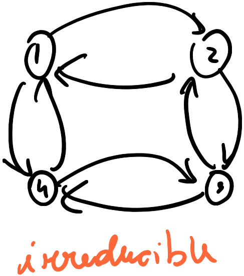

- Markov Chain *irreducible* if underlying directed graph is *strongly connected* (i.e. there is directed path from i to j for any pair $i, j \in V$)

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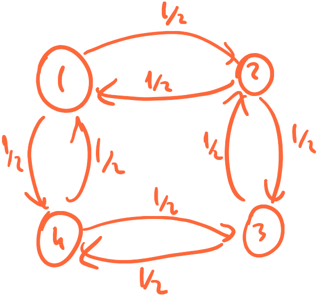

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix} \end{array}$$

- $P \in \mathbb{R}^{n \times n}$ transition matrix

$$\boxed{P_{ij} = \Pr[i \to j]}$$

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Markov chain can be seen in weighted adjacency matrix format.

if Markov chain starts at vertex/state 1

$p_0(1) = 1 \quad p_0(j) = 0 \quad \forall j > 1$

$p_0 = e_1$

$(1, 0, 0, \ldots, 0)$

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- $p_t \in \mathbb{R}^n$ probability vector: $p_t(i) :=$ Pr[being at *state i* at *time t*]

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Markov chain can be seen in weighted adjacency matrix format.

$$p_{t+1}(j) = \underbrace{\sum_{i=1}^{n} p_t(i) \cdot \underbrace{\Pr[i \to j]}_{P_{ij}}}_{(\vec{p}_t P)_j}$$

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- $p_t \in \mathbb{R}^n$ probability vector: $p_t(i) := \Pr[\text{being at } \textit{state } i \text{ at } \textit{time } t]$
- Transition given by

$$p_{t+1} = p_t \cdot P$$

Properties of Markov Chains

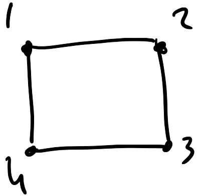
- *Period* of a state i is:

time

$$\gcd\{t \in \mathbb{N} \mid P^t_{i,i} > 0\}$$

That is, gcd of all times t such that the probability of starting at state i and being back at i at time t is positive

start at 1



even times: 1, 3

odd times: 2, 4

$\boxed{\text{Period for 1 is = 2}}$

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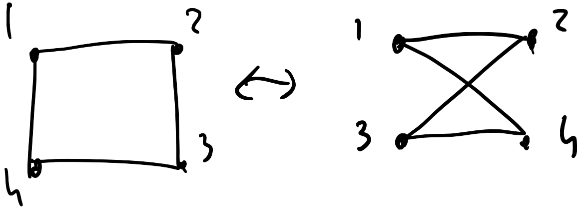
  - Bipartite graphs yield periodic Markov Chains



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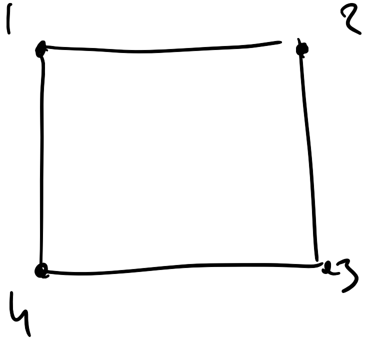
Lemma

For any finite, irreducible and aperiodic Markov Chain, there exists $\underbrace{T < \infty}_{\text{finite}}$ such that

$$P^t_{i,j} > 0 \quad \text{for any } i,j \in V \text{ and } t \geq T.$$

See proof in reference of [Häggström, Chapter 4].

Periodic Markov Chain



Periodic Markov Chain

Reducible Markov Chain

Stationary Distributions

Definition (Stationary Distribution)

A stationary distribution of a Markov Chain is a probability distribution $\pi \in \mathbb{R}^n$ such that

$$\underbrace{\pi}_{p_t} P = \underbrace{\pi}_{p_{t+1}}.$$

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- **Intuition:** If we run finite, irreducible and aperiodic Markov Chain long enough, we will converge to a stationary distribution.

unique

reducible → could have more than 1 stationary dist.

periodic → have no stationary distribution

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- Given two distributions $p, q \in \mathbb{R}^n$, their *total variational distance* is

$$\Delta_{TV}(p, q) = \frac{1}{2} \sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2} \cdot \|p - q\|_1$$

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- p_t *converges* to q iff $\lim_{t \to \infty} \Delta_{TV}(p_t, q) = 0$

Mixing Time of Markov Chains

Definition (Mixing Time)

The ε-mixing time of a Markov Chain is the smallest t such that

$$\Delta_{TV}(p_t, \pi) \leq \varepsilon$$

regardless of the initial starting distribution p_0.

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- For complete graph, eigenvalues $\boxed{\lambda_1 = 1,}$ $\boxed{\lambda_2 = \cdots = \lambda_n = -1/(n-1),}$ corresponding eigenvectors $v_1, \ldots, v_n$ (orthonormal)

$n, 0, 0, 0, \ldots, 0$

$$P = \begin{pmatrix} 0 & & & 1/_{n-1} \\ & 0 & & \\ & & 0 & \\ 1/_{n-1} & & \ddots & \\ & & & 0 \end{pmatrix} = \frac{1}{n-1} J - \frac{1}{n-1} I$$

all ones matrix

$\frac{1}{n-1}$ $n-1, -1, -1, \ldots, -1$

$(1, 1, 1, \ldots, 1) \Leftrightarrow$ eigenvalue n

$(1, -1, 0, \ldots, 0)$ eigenvalue 0

Mixing Time for Complete Graph

$$\left(\tfrac{1}{n}, \tfrac{1}{n}, \ldots, \tfrac{1}{n}\right) P = \underbrace{\left(\tfrac{1}{n}, \ldots, \tfrac{1}{n}\right)}_{v_1}$$

eigenvalue 1 corresponds to stationary distribution

$$P_t = p_0 P^t = p_0 \left(\lambda_1 v_1 v_1^T + \sum_{i=2}^{n} \lambda_i v_i v_i^T \right)^t$$

P symmetric orthonormal basis of eigenvectors

$$\boxed{v_i^T v_j = 0} \quad \forall\, i \neq j$$

stationary distribution

$$P_t = p_0 \left(\underbrace{\lambda_1^t}_{=1} \cdot v_1 v_1^T + \sum_{i=2}^{n} \lambda_i^t v_i v_i^T \right) \rightarrow (p_0 v_1) \cdot v_1^T$$

$$\lambda_i^t = \left(-\frac{1}{n-1}\right)^t \rightarrow 0$$

Mixing Time for Complete Graph

$$\boxed{p_0 = e_i}$$

$$e_i J = n \underset{\sim}{\pi} = (1/n, 1/n, \dots, \tfrac{1}{n})$$

$$p_1 = p_0 P = \frac{1}{n-1} e_i (J - I) = \frac{n}{n-1}\pi - \frac{1}{n-1} e_i$$

$$p_2 = p_1 P = \left(\frac{n}{n-1}\pi - \frac{1}{n-1}e_i\right) P = \frac{n}{n-1}\pi - \frac{1}{n-1}\left(\frac{n}{n-1}\pi - \frac{1}{n-1}e_i\right)$$

$$= \frac{n}{n-1}\pi\left(1 - \frac{1}{n-1}\right) + \frac{1}{(n-1)^2} e_i$$

$$p_t = \frac{n}{n-1}\pi \cdot \sum_{k=0}^{t-1}\left(\frac{-1}{n-1}\right)^k + \underbrace{\left(\frac{-1}{n-1}\right)^t e_i}_{\text{norm} < \epsilon}$$

$$\|p_t\|_1 = 1$$

$$\|\pi\|_1 = 1$$

$$\|e_i\|_1 = 1$$

$$p_t - \alpha_t \pi = \left(\frac{-1}{n-1}\right)^t e_i$$

$$\boxed{t = \log_{n-1}(1/\epsilon)}$$

$$p_0 = \sum_{i=1}^{n} \alpha_i \underbrace{e_i}_{} \qquad \alpha_i \in [0,1] \quad \text{s.t.} \quad \sum_{i=1}^{n} \alpha_i = 1$$

ϵ mixing time

is $\log_{n-1}(1/\epsilon)$

$\Rightarrow$ p_0 ϵ-mixes in $\log_{n-1}(1/\epsilon)$ time

(for any p_0)

Hitting Time

- Given states i, j in a Markov chain, the *hitting time* from state i to state j is defined as

$$T_{i,j} := \min\{t \geq 1 \mid X_t = j, X_0 = i\}$$

We say $T_{i,j} = \infty$ if the Markov chain never visits j starting from i.

and $T_{ij} < \infty$ otherwise

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- The *mean hitting time* $\tau_{i,j} := \mathbb{E}[T_{i,j}]$
- *Hitting time lemma*: For any *finite*, *irreducible*, *aperiodic* Markov chain, and for any two states i, j (not necessarily distinct) we have that:

$$\Pr[T_{i,j} < \infty] = 1 \quad \text{and} \quad \mathbb{E}[T_{i,j}] < \infty$$

Proof of Hitting Time Lemma

- We know that we can find $M < \infty$ such that $(P^M)_{i,j} > 0$ for all i, j, since our Markov chain is finite, irreducible and aperiodic.

(lemma from previous part)

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- set $\alpha := \min_{i,j}(P^M)_{i,j}$ = the smallest entry of P^M

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- Note that

$$\Pr[T_{i,j} > M] \leq \Pr[X_M \neq j] \leq 1 - \alpha$$

state of MC at step M

$$1 - \underbrace{\Pr[X_M = j]}_{= (P^M)_{ij} \geq \alpha}$$

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- Moreover, we can prove:

$$\begin{aligned}\Pr[T_{i,j} > 2M] &= \Pr[T_{i,j} > M] \cdot \Pr[T_{i,j} > 2M \mid T_{i,j} > M] \\ &\leq (1-\alpha) \cdot \underbrace{\Pr[X_{2M} \neq j \mid T_{i,j} > M]} \\ &\leq (1-\alpha)^2\end{aligned}$$

same reasoning as previous slide

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- Iterating, we have $\Pr[T_{i,j} > \ell M] \leq (1-\alpha)^\ell$

$\Rightarrow \Pr[T_{i,j} = \infty] = 0$

first part done

Proof of Hitting Time Lemma

$$\sum_{n\geq 0} \Pr[T_{i,j} > n] = \sum_{\ell\geq 0} \underbrace{\sum_{k=\ell M}^{(\ell+1)M-1}}_{M} \underbrace{\Pr[T_{i,j} > k]}_{\leq \Pr[T_{ij} > M\ell]}$$

$$\leq \sum_{\ell\leq 0} M\cdot \Pr[T_{ij} > \ell M] \leq \sum_{\ell\geq 0} M\cdot(1-\alpha)^{\ell}$$

$$= M\cdot\frac{1}{1-(1-\alpha)} = \frac{M}{\alpha}$$

- Iterating, we have $\Pr[T_{i,j} > \ell M] \leq (1-\alpha)^{\ell}$
- Thus, we have

$$\mathbb{E}[T_{i,j}] = \sum_{n\geq 1} \Pr[T_{i,j} \geq n] = \sum_{n\geq 0} \Pr[T_{i,j} > n] \leq M/\alpha < \infty \qquad \square$$

Fundamental Theorem of Markov Chains

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Theorem (Fundamental Theorem of Markov Chains)

Any finite, irreducible and aperiodic Markov Chain has the following properties:

1. *There exists a unique stationary distribution π, where $\pi_i > 0$ for all $i \in [n]$*
2. *The sequence of distributions $\{p_t\}_{t \geq 0}$ will converge to π, no matter what the initial distribution is*
3. $$\pi_i = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{\tau_{i,i}}$$

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2. *For every distribution* $p_0 \in \mathbb{R}^n_{\geq 0}$, $\lim_{t \to \infty} p_0 \cdot P^t = \pi$
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Intuition for proof of this theorem:

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Intuition for proof of this theorem:

- two random walks are "indistinguishable" after they "meet" at the *same vertex v* at a particular *time t*
- By finiteness, irreducibility and aperiodicity, two walks will meet with positive probability (and thus by Markov property) become *same distribution*

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If our underlying graph is undirected:

- If A_G adjacency matrix of $G(V,E)$ and $D = diag(d_1, d_2, \ldots, d_n)$, transition matrix:

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    - eigenvalue is *positive*
  - This eigenvector is π!
  - All random walks converge to π, as we wanted to show.

Revisiting the complete graph

Conclusion

Markov Chains and Random Walks are ubiquitous in randomized algorithms.

- Page Rank algorithm (next lecture)
- Approximation algorithms for counting problems [Karp, Luby & Madras]
    - Permanent of non-negative matrices [Jerrum, Vigoda & Sinclair]
- Sampling Problems
    - Gibbs sampling in statistical physics
    - many more places
- Probability amplification without too much randomness (efficient)
    - Random walks on expander graphs
- many more

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 6]
  - [Häggström]
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf`
- Also see Lap Chi's notes `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf` for a proof of fundamental theorem of Markov chains for undirected graphs.

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