Lecture 8: Sublinear Time Algorithms

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Overview

- Introduction
  - Truth and Reconciliation
  - Why Sublinear Time Algorithms?
  - Warm-up Problem
- Main Problem
  - Number of Connected Components
- Acknowledgements

National Day of Truth and Reconciliation

- Today is the day marked to honour and celebrate the national day for truth and reconciliation
- Take some time to reflect and understand the impacts of the residential school system in Canada
- Recently, over 1,308 remains of dead children were uncovered in former residential school sites
- For more information on the issues revolving around residential schools and initiatives, see:

  `https://uwaterloo.ca/indigenous/engagement-knowledge-building/events-workshops/national-day-truth-reconciliation`

How do we handle big data?

Sometimes big data does not come to us (think streaming), but instead we *can query small pieces* of it.

Sometimes big data can also *change over time*, so we need a *robust* answer and/or be able to solve problem quickly multiple times.

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  - Testing bipartiteness
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Connects to *randomized algorithms*, *approximation algorithms*, *parallel algorithms*, *complexity theory*, *statistics*, *learning*

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What we *can't* do:

- Can't answer **for all** or **there exists** or **exactly** type statements

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What we *can* do:

- Can answer **for most** or **averages** or **approximate** type statements *with high probability*
  - are most individuals connected via friendships?
  - are most individuals connected by at most 6 degrees of separation?
  - approximately how many people are left handed?
  - is my program correct on most inputs

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Randomized & *Approximate* algorithms.

Sublinear Time Models of Computation

- Random Access Queries

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    - Adjacency matrix
    - Adjacency list

input is a graph

Adjacency matrix model

$G(V,E)$ $\quad |V| = n$

$$(A_G)_{ij} = \begin{cases} 1 & \text{if there is edge } \{i,j\} \\ 0 & \text{otherwise} \end{cases}$$

input size = size of matrix = n^2

$\| $

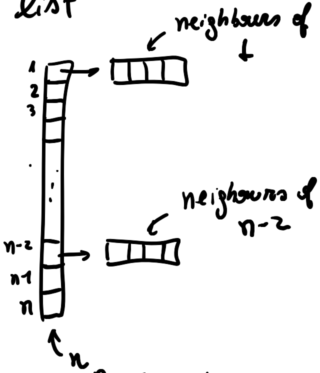
N

$\boxed{N = n^2}$

sublinear here $o(N)$

Adjacency list:

list



$$N = \sum_{i=1}^{n} (d_i + 1)$$

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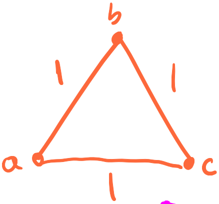
    - many others...
- Samples
  - get samples from certain distribution/input at each step

Approximate Diameter of a Point Set

- **Input:** m points and a distance matrix D such that
  - $D_{ij} \leftarrow$ distance from i to j
  - D *symmetric* and satisfies *triangle inequality*

Input given in *adjacency matrix* representation



$$D = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$\boxed{D_{ij} \leq D_{ik} + D_{kj} \qquad \forall\ i,j,k}$$

triangle inequality

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- **Output:** Indices k, ℓ such that

$$D_{k\ell} \geq D_{ab}/2$$

at least half of diameter

2-multiplicative algorithm

Algorithm & Analysis

- Pick k arbitrarily

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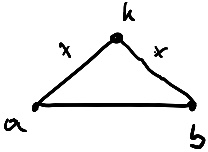

Why does this work?

- Correctness

diameter

Δ- inequality

$$\begin{aligned} D_{ab} &\leq D_{ak} + D_{kb} = D_{ka} + D_{kb} \\ &\leq D_{k\ell} + D_{k\ell} = 2 \cdot D_{k\ell} \end{aligned}$$

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- Running time: $O(m) = O(N^{1/2})$

$O(m)$

select $k \to O(1)$

need to go throug all elements D_{ki} and find max

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Is this the best we can do?

Lower Bound for Approximate Diameter

- Let D be following: distance matrix $D_{i,i} = 0, \ \forall i \in [m]$ and $D_{i,j} = 1$ otherwise

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- Let D' be same matrix as D except that for one pair (a, b) we make

$$D'_{ab} = D'_{ba} = \underline{2 - \delta}$$

any $\delta \in (0, 2)$

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- Check that D' satisfies properties of a distance matrix (thus valid)

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- Check that D' satisfies properties of a distance matrix (thus valid)
- **Practice problem:** prove that it would take $\Omega(N)$ time (i.e. number of queries) to decide if diameter is 1 or $2 - \delta$

Connected Components

How to approximate number of connected components of a graph:

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- **Input:** graph $G(V, E)$ in *adjacency list* representation. $\epsilon > 0$.

$$n = |V|, \ m = |E|, \ N = m + n$$

- **Output:** if $C \leftarrow \#$ connected components of G, output with probability $\geq 3/4$ C' such that

$$|C' - C| \leq \epsilon n$$

our estimate

actual # connected components

precision parameter

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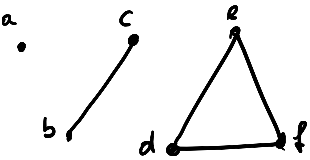
Lemma ($\#$ Connected Components)

Let $G(V, E)$ be a graph. For vertex $v \in V$, let $n_v \leftarrow \#$ vertices in connected component of v. Let C be number of connected components of G. Then:

$$C = \sum_{v \in V} \frac{1}{n_v}$$



$C = 3$

$\boxed{n_a = 1}$ $\quad$ $\boxed{n_b = n_c = \frac{1}{2}}$ 1

$\boxed{n_d = n_e = n_f = \frac{1}{3}}$ 1

$$\underbrace{1} + \underbrace{\frac{1}{2} + \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 3$$

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take linear time to compute n_v for any vertex.

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Lemma (Estimating # components)

Let

$$\boxed{n'_v = \min(n_v, 2/\epsilon)}$$

good proxies

Then

$$\left| \sum_{v \in V} \frac{1}{n_v} - \sum_{v \in V} \frac{1}{n'_v} \right| \leq \frac{\epsilon n}{2}.$$

large $\cong 2/\epsilon$

if $n_v > 2/\epsilon$

set $n'_v = 2/\epsilon$

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How do we do this estimation?

Sample vertex v and run BFS starting at v, short-cutting if see $2/\epsilon$ vertices. Cutting off BFS avoids linear time problem

Connected Components - proof of lemma

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$V = L \cup S$

large connected comp. ($> 2/\epsilon$)

small connected comp. ($\leq 2/\epsilon$)

$$\sum_{v \in V} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right) = \sum_{v \in L} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right) + \cancel{\sum_{v \in S} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right)} 0$$

$$\left| \sum_{v \in V} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right) \right| = \left| \sum_{v \in L} \left(\frac{1}{n_v} - \frac{1}{n'_v} \right) \right| \leq \sum_{v \in L} \frac{1}{n'_v} = \frac{\epsilon}{2} \cdot |L| \leq \frac{\epsilon n}{2}$$

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- Return

$$C' = \frac{n}{s} \cdot \sum_{i=1}^{s} \frac{1}{n'_{v_i}}$$

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- **Running Time:**

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  - Adding results takes $O(s) = O(1/\epsilon^2)$ time.
- Total running time $O(1/\epsilon^4)$.

$\Theta(1/\epsilon^2)$ runs

Certainly sublinear.

Algorithm - Correctness

To prove correctness we need to show that with probability $\geq 3/4$ we have

$$\left| \underbrace{\frac{n}{s} \cdot \sum_{i=1}^{s} \frac{1}{n'_{v_i}}}_{C'} - \underbrace{\sum_{v \in V} \frac{1}{n_v}}_{C} \right| \leq \epsilon n$$

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Dividing by n/s on both sides:

$$\left| \underbrace{\sum_{i=1}^{s} \frac{1}{n'_{v_i}}}_{\text{our estimate}} - \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n_v} \right| \leq \epsilon s$$

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By our previous lemma, and triangle inequality, enough to prove that w.h.p.

$$\left| \sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v} \right| \leq \frac{\epsilon s}{2}$$

Lemma and Triangle Inequality

Lemma (Estimating # components)

Let

$$n'_v = \min(n_v, 2/\epsilon)$$

Then

$$\left| \sum_{v \in V} \frac{1}{n_v} - \sum_{v \in V} \frac{1}{n'_v} \right| \le \frac{\epsilon n}{2}.$$

$$\left| \sum_{i=1}^{s} \frac{1}{n'_i} - \frac{s}{n} \sum_{i=1}^{n} \frac{1}{n_i} \right| \le \underbrace{\left| \sum_{i=1}^{s} \frac{1}{n'_i} - \frac{s}{n} \sum_{i=1}^{n} \frac{1}{n'_i} \right|}_{\le \frac{\epsilon s}{2}} +$$

$$\underbrace{\left| \frac{s}{n} \sum_{i=1}^{n} \frac{1}{n'_i} - \frac{s}{n} \sum_{i=1}^{n} \frac{1}{n_i} \right|}$$

$$= \frac{s}{n} \cdot \left| \sum \frac{1}{n_v} - \sum \frac{1}{n'_v} \right| \le \frac{s}{n} \cdot \frac{\epsilon n}{2} = \frac{s\epsilon}{2}$$

$$\le \epsilon s \quad \boxed{\text{done}}$$

Algorithm - Correctness

Want to show that with probability $\geq 3/4$:

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Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{N} X_i$. Then

$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^{N} (b_i - a_i)^2} \right)$$

Algorithm - Correctness

Want to show that with probability $\geq 3/4$:

$$\left| \sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v} \right| \leq \frac{\epsilon \cdot s}{2}$$

Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{N} X_i$. Then

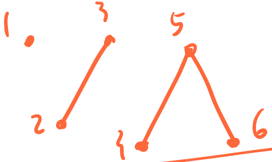
$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(- \frac{2\ell^2}{\sum_{i=1}^{N} (b_i - a_i)^2} \right)$$

Setting parameters of Hoeffing's theorem to our setting:

- $a_i = 0$, $b_i = 1$, $N = s$
- $X_i = 1/n'_v$ with probability $1/n$ (pick vertex uniformly at random)

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$



$$X_i = \begin{cases} 1 & \text{w.p. } 1/6 \\ 1/2 & \text{w.p. } 1/6 \\ 1/2 & \text{w.p. } 1/6 \\ 1/3 & 1/6 \\ 1/3 & 1/6 \\ 1/3 & 1/6 \end{cases}$$

$$\begin{array}{lll} 1 & \text{w.p.} & 1/6 \\ 1/2 & \text{w.p.} & 1/3 \\ 1/3 & \text{w.p.} & 1/2 \end{array}$$

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

$$\mu := \mathbb{E}[X] = \sum_{i=1}^{s} \mathbb{E}[X_i] = s \cdot \underbrace{\sum_{v \in V} \frac{1}{n'_v} \cdot \frac{1}{n}}_{\mathbb{E}[X_i]} = \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}$$

linearity of expectation

$$\mathbb{E}[X_i] = \sum_{v \in V} \frac{1}{n} \cdot \frac{1}{n'_v} = \frac{1}{n} \sum_{v \in V} \frac{1}{n'_v}$$

Pr[chose v]

value when choosing v

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

$$\mu := \mathbb{E}[X] = \sum_{i=1}^{s} \mathbb{E}[X_i] = s \cdot \sum_{v \in V} \frac{1}{n'_v} \cdot \frac{1}{n} = \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}$$

Hoeffding with the parameters from previous slide and $\boxed{\ell = \epsilon \cdot s/2}$:

Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[0, 1]$, $X = \sum_{i=1}^{s} X_i$. Then

$$\Pr[|X - \mu| \geq \epsilon \cdot s/2] \leq 2 \cdot \exp\left(-\epsilon^2 s/2\right) \leq 1/4$$

Algorithm - Correctness

$$X = \sum_{i=1}^{s} X_i \quad \left(= \sum_{i=1}^{s} \frac{1}{n'_{v_i}} \right)$$

$$\mu := \mathbb{E}[X] = \sum_{i=1}^{s} \mathbb{E}[X_i] = s \cdot \sum_{v \in V} \frac{1}{n'_v} \cdot \frac{1}{n} = \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}$$

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Since $s = \Theta(1/\epsilon^2)$, the result follows by choosing $s = 8 \cdot (1/\epsilon^2)$

Acknowledgement

- Lecture based largely on Ronitt's notes.
- See Ronitt's notes at http://people.csail.mit.edu/ronitt/COURSE/F20/Handouts/scribe1.pdf
- See also her notes for approximate MST http://people.csail.mit.edu/ronitt/COURSE/F20/Handouts/scribe2.pdf
- List of open problems in sublinear algorithms https://sublinear.info/index.php?title=Main_Page