Lecture 07: Algebraic Techniques
Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

September 28, 2021

Overview

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)
- Interactive proof systems (lecture 23)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)
- Interactive proof systems (lecture 23)
- Efficient proof/program verification (PCP - a bit in lecture 23)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)
- Interactive proof systems (lecture 23)
- Efficient proof/program verification (PCP - a bit in lecture 23)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs
- Cryptography

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)
- Interactive proof systems (lecture 23)
- Efficient proof/program verification (PCP - a bit in lecture 23)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs
- Cryptography
- Coding theory

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lecture 21)
- Interactive proof systems (lecture 23)
- Efficient proof/program verification (PCP - a bit in lecture 23)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs
- Cryptography
- Coding theory
- many more...

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)
- Fingerprinting for the rescue!

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)
- Fingerprinting for the rescue!

Communication complexity setting, randomized algorithms, need to work with high probability.

[1] Think of each of them being a server of a company that deals with massive data.

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$

binary rep. of a

$a_n a_{n-1} \cdots a_1$

$b_n b_{n-1} \cdots b_1$

integers are n-bit integers

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = \underbrace{x \mod p}_{\log p \text{ bits}}$ be a fingerprinting function, for a prime p

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, \underbrace{F_p(a)}_{\text{fingerprint}})$ to Bob

prime

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
    1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
    2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
   2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)

$p, F_p(a)$ $\sim 2 \log p$ bits

1 bit

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
    1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
    2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)
- if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ then protocol always right

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
    1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
    2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)
- if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ then protocol always right
- what happens when they are different?

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$

$$|M| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \qquad \alpha_i \geq 1$$

$$2^n > |M| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t} \geq \prod_{i=1}^{t} p_i > 2^t$$

$$\Rightarrow \boxed{t < n}$$

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.

$\Updownarrow$

$F_p(a) \equiv a \mod p \quad \Leftrightarrow \quad a - b \equiv 0 \mod p$

$F_p(b) \equiv b \mod p$

$0 \leq a, b \leq 2^n \Rightarrow -2^n \leq a - b \leq 2^n$

at most n primes divide $a-b$.

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $\boxed{a \neq b}$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers
- Choosing p among the first $tn\log(tn)$ ~~primes~~ integers we have that

prob. of failure

$$\Pr[F_p(a) \equiv F_p(b)] \leq \frac{\overset{\text{\# bad primes}}{n}}{\underbrace{tn \log tn / \log(tn \log tn)}_{\text{\# primes I can choose from}}} = \tilde{O}(1/t)$$

with $a \neq b$

Verifying string equality

- If $(a_1, \ldots, a_n) \neq (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$

$p \leq tn \log(tn)$

- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.

$\log p \leq \log t + \log n + \log\log tn$

- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers

$\log p = O(\log t + \log n)$

- Choosing p among the first $tn \log(tn)$ ~~primes~~ integers we have that

$$\Pr[F_p(a) \equiv F_p(b)] \leq \frac{n}{tn \log tn / \log(tn \log tn)} = \tilde{O}(1/t)$$

- Number of bits sent is $\tilde{O}(\log t + \log n)$. Choosing $t = n$ solves it.

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $\underbrace{P_1(x) \cdot P_2(x)}_{\text{this polynomial given implicitly by its two factors}} = P_3(x)$?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ arithmetic operations by FFT

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ arithmetic operations by FFT
- Polynomial evaluation: $O(n)$ arithmetic operations

(if I am given the coefficients of the polynomial)

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compute their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ arithmetic operations by FFT
- Polynomial evaluation: $O(n)$ arithmetic operations
- Can we check whether $P_1(x) \cdot P_2(x) = P_3(x)$ in $O(n)$ operations?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compute their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ arithmetic operations by FFT
- Polynomial evaluation: $O(n)$ arithmetic operations
- Can we check whether $P_1(x) \cdot P_2(x) = P_3(x)$ in $O(n)$ operations?

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$

$\deg(P_3) \leq 2n$

$\leq n$

$\deg(P_1 \cdot P_2) \leq 2n$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$. (Q)

a root of Q

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.

$S = \{1, 2, \ldots, 4n\}$

$$\Pr_{a \in_R S}\left[\underset{Q \neq 0}{Q(a) = 0}\right] = \frac{\#\text{roots of } Q \text{ in } S}{|S|} \leq \frac{2n}{4n} = \frac{1}{2}$$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$

$O(n)$

multiplication of two rational numbers

to compute $Q(a)$ in $O(n)$ arithmetic ops.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

- Can amplify probability by running multiple times or by choosing larger set S.

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ *be a field and* $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ *be a* *nonzero* *polynomial of degree* $\leq d$*. Then for any set* $S \subseteq \overline{\mathbb{F}}$*, we have:*

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid \underline{a_i \in S}] \leq \frac{d}{|S|}$$

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ be a field and $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be a nonzero polynomial of degree $\leq d$. Then for any set $S \subseteq \overline{\mathbb{F}}$, we have:

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid a_i \in S] \leq \frac{d}{|S|}$$

Proof by induction in number of variables. base case $n=1$. Induction

$$P(y, x_1, \ldots, x_n) = \sum_{i=0}^{e} P_i(x_1, \ldots, x_n)\, y^i \qquad e \leq d \qquad P_e(x_1, \ldots, x_n) \neq 0$$

$$\deg(P_e) \leq d-e$$

$S \subset \mathbb{F}$

$$\Pr_{\substack{\bar{a} \in \mathbb{F}^n \\ b \in \mathbb{F}}}[P(b, \bar{a}) = 0] \leq \underline{\Pr[P_e(\bar{a}) = 0]} \cdot \underline{\Pr[P(b, \bar{a}) = 0 \mid P_e(\bar{a}) = 0]}$$

$$+ \underbrace{\Pr[P_e(\bar{a}) \neq 0]} \cdot \Pr[P(b, \bar{a}) = 0 \mid P_e(\bar{a}) \neq 0]$$

$$\Pr\left[P(b,\bar{a})=0\right] \leq \frac{d-e}{|S|}\cdot 1 + 1\cdot\frac{e}{|S|} = \frac{d}{|S|}$$

$$\boxed{P(y,\bar{a}) = \sum_{i=0}^{e} P_i(\bar{a})\, y^i}$$

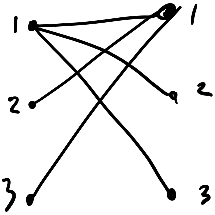
nonzero univariate polynomial of degree e

$$P(x,y) = xy^2 + y + 3 + x$$

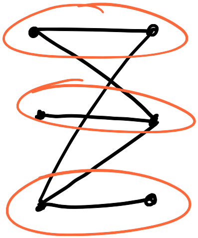
$$P(1,y) = y^2 + y + 4$$

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?



G



H

no perfect matching

$$\begin{array}{cc|ccc} & & & R & \\ & & 1 & 2 & 3 \\ \hline & 1 & 1 & 1 & 1 \\ L & 2 & 1 & 0 & 0 \\ & 3 & 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

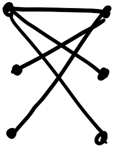
[2]First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

Symbolic adjacency matrix of G



G

$$\longleftrightarrow \quad X_G = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & 0 & 0 \\ y_{31} & 0 & 0 \end{pmatrix}$$

[2] First proved by Edmonds.

Bipartite Matching

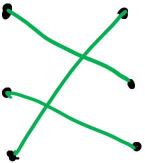
- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n\times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

monomial corresponds to a perfect matching in the graph

- $$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

sign of permutation



(123)

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

-
$$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]

$\Updownarrow$ $\Updownarrow$

there is matching $\Leftrightarrow$ monomial $(-1)^{\sigma} \prod_{i=1}^{n} y_{i\sigma(i)}$

σ s.t. $y_{i\sigma(i)}$ are in the matrix

is a non-zero monomial of $\det(X_G)$

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

- $$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$
- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]
- Testing if G has a perfect matching is a *special case* of *Polynomial Identity Testing*!

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

- $$\det(X) = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} X_{i\sigma(i)}$$

- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]
- Testing if G has a perfect matching is a *special case* of *Polynomial Identity Testing*!
- **Algorithm:** evaluate $\det(X)$ at a random value for the variables $y_{i,j}$.

[2] First proved by Edmonds.

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?
- **Tutte Matrix:** T_G is the following $2n \times 2n$ matrix: let F be an arbitrary orientation of edges in E. Then,

$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?
- **Tutte Matrix:** T_G is the following $2n \times 2n$ matrix: let F be an arbitrary orientation of edges in E. Then,

$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- $$\det(T_G) = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- $$\det(T_G) = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$$
- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{2n}$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- $$\det(T_G) = \sum_{\sigma \in S_{2n}} (-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$$
- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{2n}$.
- Each vertex in H_σ has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i)\}_{i=1}^{2n}$.
- If σ only has even cycles, then H_σ gives us a perfect matching (by taking every other edge of the graph H_σ, ignoring orientation)

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i)\}_{i=1}^{2n}$.

- Otherwise, for each $\sigma \in S_{2n}$ (that has odd cycle), there is another permutation $r(\sigma) \in S_{2n}$ that is obtained by reversing odd cycle of H_σ containing vertex with *minimum index*.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

$$G \textit{ has a perfect matching} \Leftrightarrow \det(T_G) \neq 0.$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{2n} x_{i,\sigma(i)} = -\prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{2n} x_{i,\sigma(i)} = -\prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{2n} x_{i,\sigma(i)} = -\prod_{i=1}^{2n} [T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!
- Since $r(r(\sigma)) = \sigma$, all such terms cancel!

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^\sigma \prod_{i=1}^{2n}[T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{2n}[T_G]_{i,r(\sigma)(i)}$
  - $(-1)^\sigma = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{2n}[T_G]_{i,\sigma(i)} = \prod_{i=1}^{2n} x_{i,\sigma(i)} = -\prod_{i=1}^{2n}[T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!
- Since $r(r(\sigma)) = \sigma$, all such terms cancel!
- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0.$

- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)
- If T_G has a matching, say, $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$, then take permutation $\sigma = (1\ 2)(3\ 4)\cdots(2n-1\ 2n)$

$$(-1)^\sigma \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = (-1)^n \prod_{i=1}^{n} -x^2_{(2i-1)\sigma(2i-1)} = \prod_{i=1}^{n} x^2_{(2i-1)\sigma(2i-1)}.$$

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point
- Ore-Schwartz-Zippel-deMillo-Lipton lemma tells us that this algorithm succeeds *with high probability*

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point
- Ore-Schwartz-Zippel-deMillo-Lipton lemma tells us that this algorithm succeeds *with high probability*
- We can *compute the determinant efficiently in parallel*

Great final project!

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

Think of perfect bipartite matching

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!
- Can use it to compute minimum weight perfect matching (see Lap Chi's notes)

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!
- Can use it to compute minimum weight perfect matching (see Lap Chi's notes)

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$, *if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$*.*

Example for $n = 4$:

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$*.*

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$

$\boxed{w(S_1) = 7}$ $w(S_2) = 13$ $w(S_3) = 16$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$, if we assign a random weight function $w : [n] \to [2n]$ then the probability that there is a unique minimum weight set is at least $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

$\underline{w'(S_1) = 8}$ $\underline{w'(S_2) = 8}$ $w'(S_3) = 13$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$*.*

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$, *if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is *only one* set of minimum weight.

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let

$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(\underline{B \setminus \{v\}})$$

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \overbrace{\min_{A \in \mathcal{N}_v} w(A)}^{A^*} - \overbrace{\min_{B \in \mathcal{S}_v} w(B \setminus \{v\})}^{B^*}$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set

$$w(B^*) = w(B^* \setminus \{v\}) + w(v)$$
$$= w(A^*) \underbrace{- \alpha_v + w(v)}_{\geq 0}$$
$$\geq w(A^*)$$

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.

can be computed after we set weights of $[n] \setminus \{v\}$

Proof of Isolation lemma

3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$

$\leq \sum_{i=1}^{n} \underbrace{\Pr[i \text{ amb}]}_{\leq 1/2n} \leq$

Proof of Isolation lemma

3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$
9. If two different sets A, B have minimum weight, then any element in $A \Delta B$ must be ambiguous.

in the family S

$\therefore \exists$ ambiguous

Proof of Isolation lemma

if no ambiguous element $\Rightarrow \exists$ unique minimum weight set in S

3. Let

Contrapositive

$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$

4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$
9. If two different sets A, B have minimum weight, then any element in $A \Delta B$ must be ambiguous.
10. Probability that this happens is $\leq 1/2$. (step 8)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture ~~16~~ 23)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 7]
  - [Korte & Vygen 2012, Chapter 10].
- See Lap Chi's notes at
  https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Korte, Bernhard and Vygen, Jens (2012)
Combinatorial optimization. Vol. 2. Heidelberg: Springer.

Fenner, Stephen and Gurjar, Rohit and Thierauf, Thomas (2019)
Bipartite perfect matching is in quasi-NC.
SIAM Journal on Computing

Svensson, Ola and Jakub Tarnawski (2017)
The matching problem in general graphs is in quasi-NC.
IEEE 58th Annual Symposium on Foundations of Computer Science