Lecture 6: Graph Sparsification

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September 23, 2021

Overview

- Introduction
  - Why Sparsify?
  - Warm-up Problem
- Main Problem
  - Graph Sparsification
- Acknowledgements

Why do we sparsify?

Often times graph algorithms for graphs $G(V, E)$ have runtimes which depend on $|E|$. If the graph is dense, i.e. $\underline{|E| = \omega(n^{1+c})}$ then this may be *too slow*.

$c > 0$ positive

We want graph that has *nearly-linear* number of edges $\underline{O(n \cdot \operatorname{poly}\log n)}$

$n \log^{\gamma} n$

$\gamma > 0$

$\tilde{O}(n)$

- Settle for *approximate answers*

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Graph "sparse"

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- Used as primitives in many other algorithms (for instance, max-flow, sparsest cut, etc.)

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- Settle for *approximate answers*
- Used as primitives in many other algorithms (for instance, max-flow, sparsest cut, etc.)
- Applications in network connectivity

Question: what properties of the original graph will the sparse graph retain?

Graph Cuts

Definition (Graph Cut)

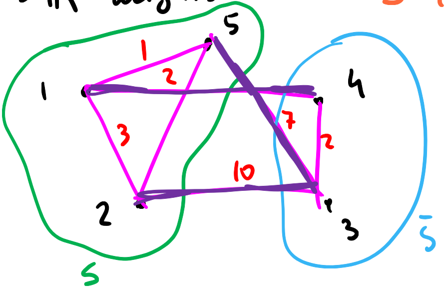
If $G(V, E, w)$ is a weighted graph, a *cut* is a partition of the vertices into two non-empty sets $V = S \sqcup \overline{S}$. The *value* of a cut is the quantity

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w_e.$$

$w : E \hookrightarrow \mathbb{R}$ weights

$S = \{1, 2, 5\}$

$\overline{S} = \{3, 4\}$



$w(S, \overline{S}) = w(3,5)$

$+ w(2,3) + w(1,4)$

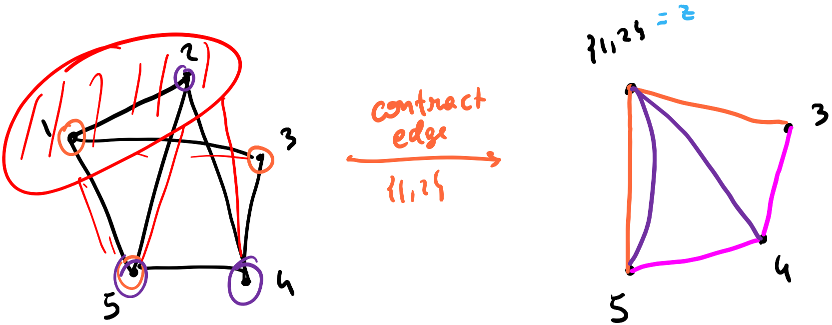
$= 7 + 10 + 2 = 19.$

Contraction of Edges

Definition (Edge Contraction)

Let $G(V,E)$ be a graph. If $e = \{u,v\} \in E$ is an edge of G, then the *contraction* of e is a new graph $H(V \cup \underline{\{z\}} \setminus \underline{\{u,v\}}, F)$ where we replace the vertices u, v by *one* vertex z, and any edge $\{u,x\} =: f \in E \setminus \{e\}$ is replaced by $\{z,x\} \in F$.

$\{v, x\}$



Randomized Minimum Cut

- **Input:** undirected unweighted graph $G(V, E)$
- **Output:** minimum cut $(S, \overline{S})$, with high probability

Definition : (min-cut)

$$\text{min cut} = \min_{S \subset V} w(S, \bar{S})$$

Randomized Minimum Cut

- **Input:** undirected unweighted graph $G(V, E)$
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- While there are more than 2 vertices in the graph:
  - Pick uniformly random edge and contract it

Randomized Minimum Cut

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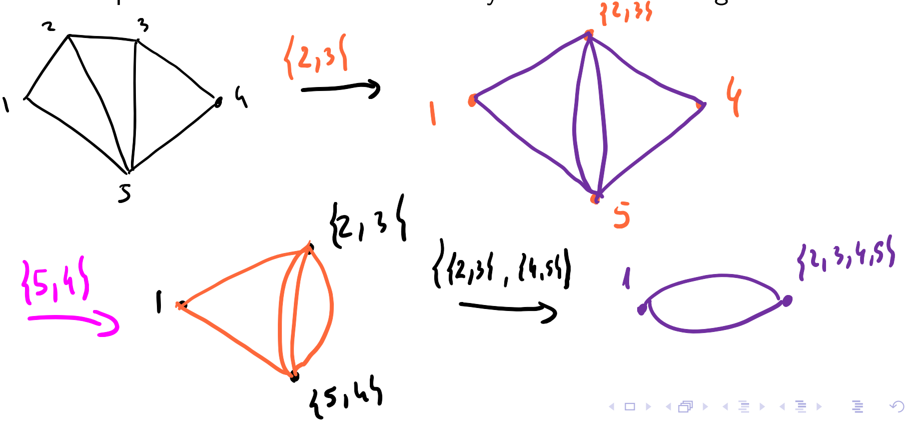
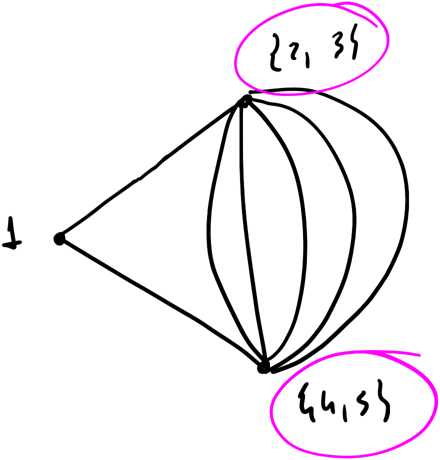
- While there are more than 2 vertices in the graph:
  - Pick uniformly random edge and contract it
- Output the two subsets encoded by the two remaining vertices.

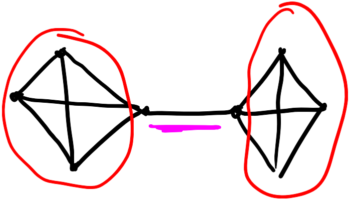


Analysis

Why does this work? (with high probability w.h.p.)

Intuition: picking a random edge uniformly at random "favours" *small cuts* (i.e. preserves them) with higher probability.





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Remark

The value of the minimum cut only increases or stays the same after contraction.

why contraction is good operation
when looking for cuts

Contraction is simulating picking a cut.

Analysis

Theorem (Karger)

The probability that the algorithm outputs a minimum cut is at least $2/n(n-1)$*, where* $n = |V|$*.*

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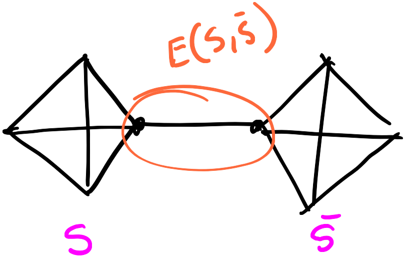
- Let $(S, \overline{S})$ be a minimum cut, and $k := |E(S, \overline{S})|$. If we never contract an edge from $E(S, \overline{S})$, the algorithm succeeds.



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- Probability that an edge from $E(S, \overline{S})$ is contracted in the i^{th} iteration (conditioned on cut still alive)
  - Each vertex is a cut, so each vertex has degree $\geq k \Rightarrow$

$$\geq \frac{(n-i+1)\cdot k}{2} \quad \text{edges remain.}$$

$$2\left(\text{\# edges in my current graph}\right) \geq \underbrace{(\text{\# vertices})}_{n - \underbrace{\text{\# contracted vertices}}_{(i-1)}} \cdot \underbrace{\text{min-degree}}_{\geq k}$$

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  - Contracting random edge, probability we kill cut $(S, \overline{S})$ is

    $$= \boxed{|E(S,\overline{S})|} \cdot \frac{1}{(\#\text{ edges})} \leq k \cdot \frac{2}{(n-i+1)\cdot k} = \underbrace{\frac{2}{n-i+1}}_{\text{upper bd on}}$$

Pr. that kill cut $(S,\overline{S})$ i^{th}

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- Probability that an edge from $E(S, \overline{S})$ is contracted in the i^{th} iteration (conditioned on cut still alive)
  - Each vertex is a cut, so each vertex has degree $\geq k \Rightarrow$

$$\Pr\left[(S,\bar{S}) \text{ survives } i\text{th iteration} \;\middle|\; \text{alive at } i\text{-}1\right] \geq 1 - 2/n\text{-}i\text{+}1$$

$$\geq \frac{(n-i+1)\cdot k}{2} \quad \text{edges remain.}$$

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  - $\Pr[(S,\overline{S}) \text{ survives}] \geq \underset{\text{1st iteration}}{(1-2/n)} \cdot (1-\underset{2/n\text{-}1}{\cancel{3/n}}) \cdots (1-2/3) = 2/n(n-1)$

Hmmmmm, this is not with high probability...

- To improve success probability, repeat this randomized procedure t times (for which t?)
- If we repeat for t times, failure probability is

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^t$$

$$\Pr[\text{success}] \geq \frac{2}{n(n-1)}$$

$$\Pr[\text{fail}] \leq 1 - \frac{2}{n(n-1)}$$

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- setting $t = 2n(n-1)$ then

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^t \leq \exp\left(-\frac{2t}{n(n-1)}\right) = \underline{e^{-4}}$$

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- **Running time:** One execution implemented in $O(n^2)$, so t executions in time $O(n^2 t) = O(n^4)$.
- You will work on some running time improvements in your homework!

Combinatorial Application

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There are at most $O(n^2)$ *minimum cuts in an undirected graph.*

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There are at most $O(n^2)$ minimum cuts in an undirected graph.

- Each minimum cut survives with probability $\Omega(1/n^2)$
- Events that two different cuts survive are disjoint

$C_1, C_2, \dots, C_t$

$$\sum_{i=1}^{t} \underbrace{\Pr[C_i \text{ survives}]}_{\geq 2/n(n-1)} \leq 1 \qquad \Rightarrow \frac{2t}{n(n-1)} \leq 1$$

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This is all good, but we haven't "sparsified" anything so far!

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Graph Sparsification

Definition (Weight of a cut)

Let $G(V, E, w)$ be undirected weighted graph. For any cut $(S, \overline{S})$, let the weight of $(S, \overline{S})$ be

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w(e).$$

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Definition (Sparse Graph)

We say that a graph $G(V, E)$ is *sparse* if $|E| = \tilde{O}(|V|)$.

$= O(|V| \cdot \text{polylog}\, |V|)$

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Question

How to make a graph sparse (nearly linear # edges) while approximating the value of every cut of a graph?

Graph Sparsification

- **Input:** graph $G(V, E, w_G)$, $\varepsilon > 0$.

$$n = |V|, \;\; m = |E|.$$

- **Output:** graph $H(\underline{V}, \underline{F}, \underline{w_H})$ such that *for every cut $(S, \overline{S})$*, we have

$$(1-\varepsilon)\cdot w_G(S,\overline{S}) \le \underbrace{w_H(S,\overline{S})}_{\substack{\text{weight}\\ \text{of cut}\\ (S,\overline{S})\\ \text{in } H}} \le (1+\varepsilon)\cdot w_G(S,\overline{S})$$

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- *Assumption (for this class):* the input graph $G(V, E)$ is unweighted and has minimum cut value $\Omega(\log n)$ (i.e., a large-ish cut)

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Algorithm:

- Let $p \in (0, 1)$ be a parameter.
- For each edge $e \in E(G)$, with probability p, make e an edge of H with weight $w_H(e) = 1/p$.

Graph Sparsification

Idea:

- Set p to get correct expected value for both $\#$ edges in H and the value of each cut $(S, \overline{S})$ in H.

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- Set p to get correct expected value for both # edges in H and the value of each cut $(S, \overline{S})$ in H.
- After that, need to prove concentration around expected values *for all cuts simultaneously*!

2^{n-1} cuts!

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Theorem ([Karger, 1993])

Let c be the value of the min-cut of G. Set

$$p = \frac{15 \ln n}{\varepsilon^2 \cdot c}.$$

Graph H given by algorithm from previous slide approximates all cuts of G *and has $O(p \cdot |E|)$ edges with probability $\geq 1 - 4/n$.*

Graph Sparsification

- Take a cut $(S, \overline{S})$. Suppose $k := w_G(S, \overline{S})$. Let

  $$X_e = \begin{cases} 1, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$

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-

$$\mathbb{E}[\underbrace{|F|}_{\text{set of edges in } H}] = \sum_{e\in E} \underline{\mathbb{E}[X_e]} = \sum_{e\in E} (\underline{p \cdot 1 + (1-p) \cdot 0}) = p \cdot |E|$$

$$|F| = \sum_{e \in E} X_e$$

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$$w_H(e) = \begin{cases} 1/p & w.p.\ p \\ 0 & w.p\ 1-p \end{cases}$$

- $$\mathbb{E}[|F|] = \sum_{e \in E} \mathbb{E}[X_e] = \sum_{e \in E} (p \cdot 1 + (1-p) \cdot 0) = p \cdot |E|$$

- $$\mathbb{E}[w_H(S, \overline{S})] = \sum_{e \in E(S,\overline{S})} \mathbb{E}[\underline{w_H(e)}] = \sum_{e \in E(S,\overline{S})} \left(p \cdot \frac{1}{p} + (1-p) \cdot 0\right)$$
$$= \underline{|E(S, \overline{S})| = k = w_G(S, \overline{S})}$$

original graph is unweighted

$$w_H(S, \bar{S}) = \sum_{e \in E(S,\bar{S})} w_H(e)$$

Graph Sparsification - Concentration

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  $$w_e = \begin{cases} 1/p, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$
- $w_H(S, \overline{S})$ is a sum of independent random variables w_e

Graph Sparsification - Concentration

- Take a cut $(S, \overline{S})$. Suppose $\underline{k := w_G(S, \overline{S})}$. Let $= \mathbb{E}[w_H(S,\overline{S})]$

  $$w_e = \begin{cases} 1/p, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$

- $w_H(S, \overline{S})$ is a sum of independent random variables w_e
- Chernoff Bound:

$$\Pr[|\underline{w_H(S,\overline{S})} - \underline{k}| \geq \varepsilon \cdot k] \leq 2\exp\left(-\frac{\varepsilon^2 kp}{3}\right) = 2n^{-5k/c}$$

expectation

using

$$p = \frac{15 \ln n}{\varepsilon^2 c}$$

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- This is probability of *single cut* deviating from its mean... How can we handle the *exponentially many* cuts in the graph?

2^{n-1}

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- **Observation:** probability that large cut violated is *much smaller*, and there are *not many small cuts*!
- So we can do a clever union bound!

Number of Cuts Lemma

c = weight of min cut

Lemma (Number of small cuts)

The number of cuts with at most $\alpha \cdot c$ edges for $\alpha \geq 1$ is at most $n^{2\alpha}$.

Practice : prove this !

Number of Cuts Lemma

Lemma (Number of small cuts)

The number of cuts with at most $\alpha \cdot c$ *edges for* $\alpha \geq 1$ *is at most* $n^{2\alpha}$.

Practice problem: generalize our earlier proof on the # minimum cuts to this case.

Union Bound on # Cuts

$$\Pr[\text{some cut is violated}] \overset{\text{union bound}}{\le} \sum_{\underbrace{S \subseteq V}_{\text{over all cuts}}} \Pr[(S, \overline{S}) \text{ is violated}]$$

Union Bound on # Cuts

$$\Pr[\text{some cut is violated}] \le \sum_{S\subseteq V} \Pr[(S,\overline{S}) \text{ is violated}]$$

$$\le \sum_{\alpha=1,2,4,8,\ldots} \boxed{\underbrace{\sum_{\substack{S\subseteq V \\ \alpha c \le |w_G(S,\overline{S})| \le 2\cdot\alpha c}}}_{\text{grouping all cuts of weight between } \alpha c \text{ and } 2\alpha c} \Pr[(S,\overline{S}) \text{ is violated}]}$$

Q: how many cuts in grouping above?

A: by our lemma $\le n^{4\alpha} = n^{2\cdot(2\alpha)}$

Union Bound on # Cuts

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$$\le \sum_{\alpha=1,2,4,8,\ldots} \boxed{\sum_{\substack{S \subseteq V \\ \alpha c \le |w_G(S,\overline{S})| \le 2\cdot\alpha c}}} \underline{\Pr[(S, \overline{S}) \text{ is violated}]} \swarrow$$

→ has $\le n^{4\alpha}$ summands

$$\le \sum_{\alpha=1,2,4,8,\ldots} \underbrace{\underline{n^{4\alpha}} \cdot \Pr[(S, \overline{S}) \text{ is violated} \mid \underline{\alpha c} \le \underbrace{|w_G(S, \overline{S})|}_{k} \le 2 \cdot \alpha c]}_{\le 2n^{-5k/c} \le 2n^{-5\alpha c/c}}$$

$$= 2n^{-5\alpha}$$

Union Bound on # Cuts

$$\Pr[\text{some cut is violated}] \le \sum_{S\subseteq V} \Pr[(S,\overline{S}) \text{ is violated}]$$

$$\le \sum_{\alpha=1,2,4,8,\ldots} \sum_{\substack{S\subseteq V\\ \alpha c\le |w_G(S,\overline{S})|\le 2\cdot\alpha c}} \Pr[(S,\overline{S}) \text{ is violated}]$$

$$\le \sum_{\alpha=1,2,4,8,\ldots} n^{4\alpha}\cdot \underbrace{\Pr[(S,\overline{S}) \text{ is violated} \mid \alpha c \le |w_G(S,\overline{S})| \le 2\cdot \alpha c]}$$

$$\le \sum_{\alpha=1,2,4,8,\ldots} \underline{n^{4\alpha}}\cdot \underline{2n^{-5\alpha c/c}} \rightarrow 2n^{-5\alpha}$$

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Another application of Chernoff gives us that H has the right number of edges $|F| \approx p \cdot |E|$ (i.e., sparse)

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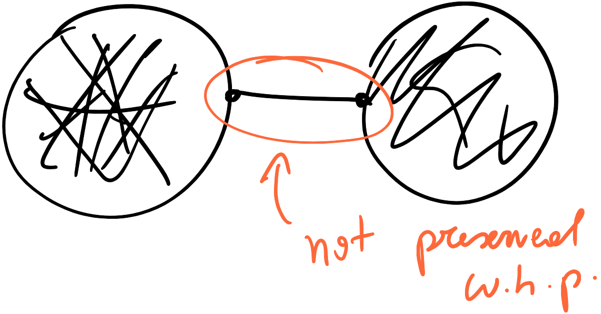

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- Sample edge e with probability $\boxed{p_e = \Theta\left(\frac{\log n}{\varepsilon^2 \cdot s_e}\right)}$ and weight $1/p_e$.

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's Lecture 1 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L01.pdf`
- See Lap Chi's Lecture 3 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L03.pdf`
- See Mohsen's notes for the general Benczur-Karger algorithm `https://people.inf.ethz.ch/gmohsen/AA18/Notes/S1.pdf`.

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