Lecture 5: Hashing

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Overview

- Introduction
  - Hash Functions
  - Why is hashing?
  - How to hash?
- Succinctness of Hash Functions
  - Coping with randomness
  - Universal Hashing
  - Hashing using 2-universal families
  - Perfect Hashing
- Acknowledgements

Computational Model

Before we talk about hash functions, we need to state our model of computation:

Definition (Word RAM model)

In the word RAM[a] model:

- all elements are integers that fit in a machine word of w bits
- Basic operations (comparison, arithmetic, bitwise) on such words take $\Theta(1)$ time
- We can also access *any* position in the array in $\Theta(1)$ time

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Wait, but aren't we working on *asymptotic analysis* of algorithms?
Yes, but this model is still relevant for problems of good enough size (so asymptotics can kick in) but not super huge that words don't fit in a machine word.

What is hashing?

Store $O(n)$ elements (keys) from the set $U = \{0, 1, \ldots, m-1\}$,[1] where $m >> n$, in a data structure that supports *insertions, deletions, search* "as efficiently as possible."

[1] Here we assume that m smaller than our memory. So $\log m \leq w$.

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- Insertion: $O(1)$, Deletion: $O(1)$, Search: $O(1)$ wonderful
- Memory: $O(m)$ *(this is very bad!)*

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- Memory: $O(m)$ *(this is very bad!)*

Want to also achieve optimal memory $O(n)$. For this we will use a technique called *hashing*.

- A *hash function* is a function $h : U \to [0, n-1]$, where $|U| = m >> n$.
- A *hash table* is a data structure that consists of:
  - a table T with n cells $[0, n-1]$, each cell storing a word
  - a hash function $h : U \to [0, n-1]$

From now on, we will define memory as *# of cells*.

[1] Here we assume that m smaller than our memory. So $\log m \leq w$.

Why is hashing useful?

- Designing efficient data structures (dictionaries) for searching
- Data streaming algorithms
- Derandomization
- Cryptography
- Complexity Theory
- many more

Challenges in Hashing

Setup:

- Universe $U = \{0, \ldots, m-1\}$ of size $m >> n$ where n is the size of the range of our hash function $h : U \rightarrow [0, n-1]$
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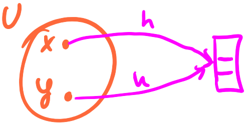
Ideally, want hash function to map *different keys* into *different locations*.

Definition (Collision)

We say that a *collision* happens for hash function h with inputs $x, y \in U$ if $x \neq y$ and $h(x) = h(y)$.

By pigeonhole principle, impossible to achieve no collisions without knowing keys in advance ($|U| >> n$).

collision:



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Will settle for: # collisions *small with high probability*.

Our solution: family of hash functions

Construct *family* of hash functions $\mathcal{H}$ such that the *number of collisions* is **small** with **high probability**, when we pick hash function uniformly at random from the family $\mathcal{H}$.

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Simplest version to keep in mind:

$$\Pr_{h \in_R \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{\text{poly}(n)} \qquad \forall x \neq y \in U$$

Prob. when picking h unif. at random from $\mathcal{H}$

they collide

any pair of keys

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Simplest version to keep in mind:

$$\Pr_{h \in_R \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{\text{poly}(n)} \qquad \forall x \neq y \in U$$

Assumptions:

- keys are independent from hash function we choose.
- we **do not** know keys in advance (even if we did, nontrivial problem!)

Question

Still could have collisions. How do we handle them?

Random Hash Functions?

Natural to consider following approach:

From all functions $h : U \to [0, n-1]$, just pick one uniformly at random.

$:= \{0, 1, 2, \ldots, n-1\}$

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This setting is same as our balls-and-bins setting!

U set of balls

$[0, n-1]$ set of bins

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So, if we have to store n keys:

- Expected number of keys in a location: 1
- maximum number of collisions (max load) in one particular location: $O(\log n / \log \log n)$ keys

typical

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Solving collisions: store all keys hashed into location i by a linked list.

Known as *chain hashing*.

Insert $O(1)$ search / deletion $O(\log n / \log \log n)$ whp

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Could also pick *two* random hash functions and use *power of two choices*. Collision bound becomes $O(\log \log n)$.

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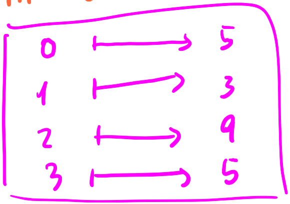
- Storing entire function $h : U \to [0, n-1]$ require $O(m)$ cells (way too much space!)
- Even if we only stored the elements we saw, would require $O(n)$ time to evaluate $h(x)$ (need to decide if we had already computed it!)

$m = 10000$ $\quad$ $n = 10$



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How do we cope with the computational problem that arose with randomness?

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How to cope with "hardness" of randomness?

We want something that is *random-like* (few collisions w.h.p.) but *easy to compute/represent*.

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$O(\log m)$ time (bit complexity)

Question

How many hash functions can we have with the property above?

function better have description ("code length")

$O(\log m)$ bits

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poly(m) functions, as each function takes at most $O(\log m)$ bits to describe. Thus these are *succinct functions* (easy to describe and compute) which have *random-like* properties!

$= 2^{O(\log m)} = 2^{c \cdot \log m} = m^c$

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Part of *derandomization/pseudorandomness*: huge subfield in TCS!

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k-wise independence

Weaker notion of independence.

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Definition (Full Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be (fully) independent if for any subset $J \subseteq [n]$ they satisfy

$$\Pr\left[\bigcap_{i \in J} X_i = a_i\right] = \prod_{i \in J} \Pr[X_i = a_i]$$

any values a_i $i \in J$

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Definition (k-wise Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be k-wise independent if for any set $J \subseteq [n]$ such that $|J| \leq k$ they satisfy

small

$$\Pr\left[\bigcap_{i \in J} X_i = a_i\right] = \prod_{i \in J} \Pr[X_i = a_i]$$

Pairwise independence

When $k = 2$, k-wise independence is called *pairwise independence*.

independent

Example (XOR pairwise independence)

Given b uniformly random bits $Y_1, \ldots, Y_b$, we can generate $2^b - 1$ uniformly distributed pairwise independent random variables as follows:

$$X_S := \bigoplus_{i \in S} Y_i \qquad S \subseteq [b], \quad S \neq \emptyset$$

$X_{\{1,2,3\}} = Y_1 \oplus Y_2 \oplus Y_3$

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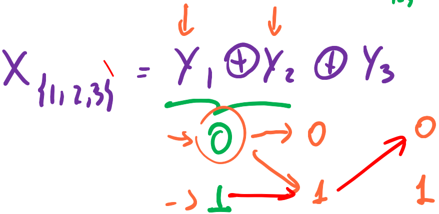

$X_{123} = 0 \leftarrow \Pr_{Y_1,Y_2,Y_3}[Y_3 = Y_1 \oplus Y_2] = 1/2$

$\Pr[Y_3 \neq Y_1 \oplus Y_2] = 1/2 \rightarrow X_{123} = 1$

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- Why are they even random?
- Why are they pairwise independent?

$S \neq T \quad \exists\, i \in S \setminus T$

Y_i independent of $Y_j \quad j \in T$

X_S X_T

$$\Pr[X_S = b_1 \text{ and } X_T = b_2] = \Pr[X_T = b_2] \cdot \Pr[X_S = b_1 \mid X_T = b_2] \to \Pr[X_S = b_1]$$

$Y_i \qquad Y_j = a_j$

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- Why are they even random?
- Why are they pairwise independent?
- Are they also 3-wise independent?

NO : $X_{\{1\}}$ $X_{\{2\}}$ $X_{\{1,2\}}$

1 1 0

Pairwise independence II

Example (Pairwise independence in $\mathbb{F}_p$)

set integers mod p

Let p be a prime number. Given 2 uniformly random variables $Y_1, Y_2 \sim [0, \ldots, p-1]$, generate p pairwise independent random variables as follows:

$\mathbb{F}_p$

$$X_i := Y_1 + i \cdot Y_2 \mod p \qquad i \in [0, p-1]$$

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fixing Y_2 Y_1 still uniform

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- Why are they even random? $\Pr[X_i = a] = 1/p$
- Why are they pairwise independent?

$X_i = a$, $X_j = b$

$$\left.\begin{aligned} Y_1 + i\,Y_2 &= a \\ Y_1 + j\,Y_2 &= b \end{aligned}\right\} \text{ exactly 1 solution } (Y_1, Y_2)$$

$$\Pr[X_i = a, X_j = b] = \frac{1}{p^2} = \Pr[X_i = a] \cdot \Pr[X_j = b]$$

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No : X_0 X_1 X_2

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- Why are they even random?
- Why are they pairwise independent?
- Are they also 3-wise independent?

Can think of these random variables as picking a random line over a finite field. If we only know one point of the line, the second point is still uniformly random. However two points determine the line.

Universal Hash Functions

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Definition (Universal Hash Functions)

Let U be a universe with $|U| \geq n$. A family of hash functions $\mathcal{H} = \{h : U \to [0, n-1]\}$ is *k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$, we have

$$\Pr_{h \in_R \mathcal{H}}[h(u_1) = h(u_2) = \ldots = h(u_k)] \leq 1/n^{k-1}$$

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Definition (Strongly Universal Hash Functions)

$\mathcal{H} = \{h : U \to [0, n-1]\}$ is *strongly k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$ and *for any* values $y_1, \ldots, y_k \in [0, n-1]$, we have

$$\Pr_{h \in_R \mathcal{H}} [h(u_1) = y_1, \ldots, h(u_k) = y_k] = 1/n^k$$

Relation to k-wise independent random variables

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Family $\mathcal{H}$ is *strongly k-universal* if the random variables $h(0), \ldots, h(|U| - 1)$ are *k-wise independent*.

Relation to k-wise independent random variables

What do the previous definitions have to do with random variables?

Family $\mathcal{H}$ is *strongly k-universal* if the random variables $h(0), \ldots, h(|U| - 1)$ are *k-wise independent*.

Can use random variables to construct universal hash functions!

Strongly 2-universal families of hash functions

Let p be a prime number, $U = [0, p-1]$.

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := a \cdot x + b \mod p \quad | \quad a, b \in [0, p-1]\}$$

is strongly 2-universal.

$h : U \to U$

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$\mathbb{F}_{p^k} \equiv \mathbb{F}_p^k$ $n = p$ $[0, p-1]$

Proposition

Let $U = [0, p^k - 1] \equiv [0, p-1]^k$ *and* $\vec{a} = (a_0, \ldots a_{k-1})$

$$\mathcal{H} = \{h_{\vec{a},b}(\vec{x}) := \vec{a} \cdot \vec{x} + b \mod p \mid \vec{a} \in U, b \in [0, p-1]\}$$

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Strongly 2-universal families of hash functions

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2-universal families of hash functions

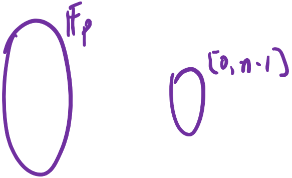
What if my hash table size is not a prime?

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := (a \cdot x + b \mod p) \mod n \mid a, b \in [0, p-1], a \neq 0\}$$

is 2-universal (but not strongly 2-universal).

Practice problem: prove the proposition above.



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- YES! Instead of constructing random lines (degree 1 polynomials), can construct random univariate polynomials of degree $k - 1$

$$y_1 + t y_2 + t^2 y_3$$

3-wise independent

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- YES! Instead of constructing random lines (degree 1 polynomials), can construct random univariate polynomials of degree $k - 1$
- Two points determine a line. Similarly, k points determine a univariate polynomial of degree $k - 1$

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- Two points determine a line. Similarly, k points determine a univariate polynomial of degree $k-1$
- Random degree $k-1$ polynomials are k-wise independent!
- Practice problem: prove this!

Efficiency

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Remark

- In XOR example, our function takes $O(b)$ storage space, and $O(b)$ time to compute.[a]
- In $\mathbb{F}_p$ examples, our function takes $O(1)$ storage space and $O(1)$ time to compute![b]

[a] Reminder that we assume that $b < w$.
[b] We assume that $p < 2^w$.

$O(\log n)$ input $S \subset [b]$ $n = 2^b - 1$

$h_{a,b}(x) = ax + b \mod p$

store a, b

Hashing with 2-universal families

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$$X_{ij} = \begin{cases} 1 & \text{if keys } i, j \text{ map to same cell} \\ 0 & \text{otherwise} \end{cases} \qquad \Pr[X_{ij}=1] \leq \frac{1}{n}$$

$$X = \sum_{i<j} X_{ij} \quad \# \text{ collisions}$$

$$\mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] \leq \sum_{i<j} \frac{1}{n} \leq \frac{\ell^2}{2n}$$

By Markov $\Pr\left[X > \ell^2/n\right] < \frac{1}{2}$

many collisions

Hashing with 2-universal families

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Thus, by Markov's inequality, we have

Lemma (Maximum load of entry of hash table)

With probability $\geq 1/2$ the maximum load when inserting ℓ elements in a table of size n using a 2-universal hash family is

$$\leq \sqrt{\frac{2\ell^2}{n}}.$$

When $\ell \approx n$ (as is usually assumed in hashing), we expect $\sqrt{2n}$.

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- Let C be the ~~number of collisions in~~ maximum load a cell

$$\binom{C}{2} \leq X \Rightarrow \Pr\left[C \geq \sqrt{\frac{2\ell^2}{n}}\right] \leq 1/2$$

$\binom{C}{2}$: # collisions coming from that cell

X: total # collisions

$\Pr\left[\frac{C^2}{2} > \frac{\ell^2}{n}\right] < \frac{1}{2}$ ($C > \sqrt{\frac{2\ell^2}{n}}$)

$\Pr\left[\frac{C^2}{2} > t\right] < \Pr[X > t]$

Markov: $\Pr\left[X > \frac{\ell^2}{n}\right] < 1/2$

Perfect Hashing

Setup: the set of keys is *static* (i.e., we know them in advance).
How to build a hash table with $O(1)$ search time and $O(n)$ memory? Can we still do it with a 2-universal family of hash functions?

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Corollary

If $h \in \mathcal{H}$ is a random hash function from a 2-universal family of hash functions, then for any set $S \subseteq U$ of size $\ell \leq \sqrt{n}$, the probability of h being perfect for S is at least $1/2$.

Proof: There is no collision with probability $\geq 1/2$.

$$\Pr\left[\text{max load} \geq \sqrt{\frac{2\ell^2}{n}}\right] \leq 1/2$$

$$\ell \leq \sqrt{n} \Rightarrow \sqrt{\frac{2\ell^2}{n}} \leq \sqrt{2} < 2$$

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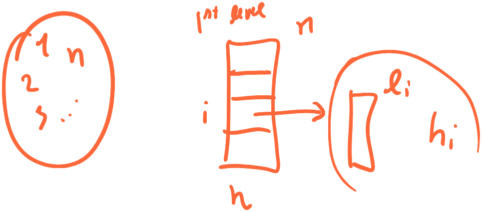
New idea: build a *two-level* hash table!



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Theorem

The two-level approach gives perfect hashing scheme.

have $O(n)$ hash tables memory $O(n)$

Proof of Theorem

S set of keys

Approach: pick first layer hash function h uniformly at random. Test h on S.

With probability $\geq \frac{1}{2}$ max load in a cell is $\leq \sqrt{n}$. We know that get good h in constant # tries.

Assume that max load (h, S) is $\leq \sqrt{n}$

$l_i \leftarrow$ load at i^{th} cell from (h, S)

Proof of Theorem

know: $l_i \leq \sqrt{n}$ (because h is good)

also know $\sum_{i=1}^{n} l_i = n = |S|$

Corollary if take h_i random from 2-universal family $h_i : S \to l_i^2$ h_i is perfect for the l_i elements mapping to ith cell. (w.h.p.)

expected # collisions $= 0$ this scheme is perfect

Bound on memory: $n + \sum_{i=0}^{n-1} l_i^2$

Proof of Theorem

$$\sum_{i=0}^{n-1} l_i^2 \leq 2(\#\text{ collisions in hash function } h)$$

collisions for 1^{st} hash function is w.h.p.

$$\leq \frac{n^2}{2n} = \frac{n}{2}$$

$$\sum_{i=0}^{n-1} l_i^2 = O(n) \implies \text{memory is } O(n)$$

Proof of Theorem

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's notes at
  https://cs.uwaterloo.ca/~lapchi/cs466/notes/L05.pdf

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