Lecture 4: Balls & Bins

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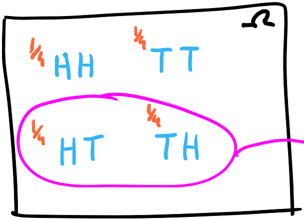
Overview

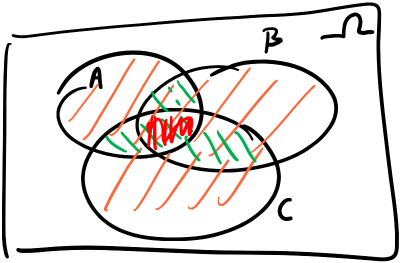
Event Spaces and Inclusion-Exclusion

Ω - event space (outcomes of tossing fair coin twice unbiased/independent)



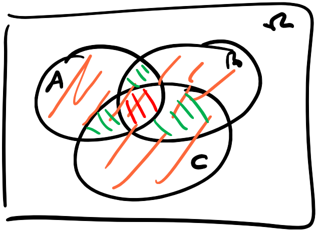
event : exactly 1 H come up

orange region: $A \cup B \cup C$



$$\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C] + \Pr[A \cap B \cap C]$$

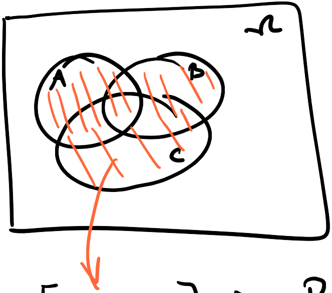
Union Bound and Inclusion-Exclusion

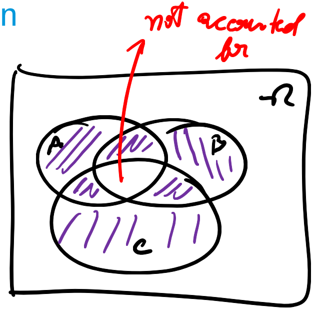


$$\Pr[\text{at least one } A, B \text{ or } C \text{ happens}] \stackrel{\Delta}{=} \Pr[A \cup B \cup C]$$

$$\leq \Pr[A] + \Pr[B] + \Pr[C]$$

Union Bound and Inclusion-Exclusion



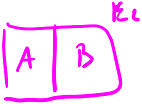


$$\Pr[A \cup B \cup C] \geq \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[A \cap C] - \Pr[B \cap C]$$

we can generalize this to any level of intersections

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is



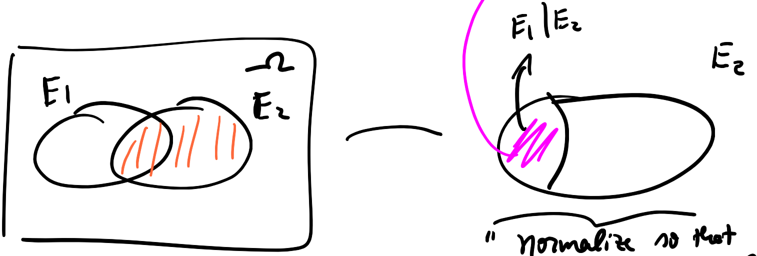
$$\Pr[E_1 \mid E_2] := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$

$\Pr[E_2] = \Pr[A] + \Pr[B]$

$\Pr[A \mid E_2] = \frac{\Pr[A]}{\Pr[E_2]}$

Probability of E_1 happens given that event E_2 happened

both E_1 and E_2 happens



" normalize so that E_2 has" Pr 1

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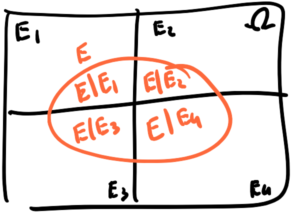
- If $E_1, \ldots, E_k$ partition our sample space, then for any event E

$$\Pr[E] = \sum_{i=1}^{k} \Pr[E \mid E_i] \cdot \Pr[E_i]$$

$\Pr[E \cap E_i]$



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- **Simple Bayes' rule**:

Pr[E1 ∩ E2]

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- **Bayes' rule**: $E_1, \ldots, E_k$ partition our sample space then for event E

$$\Pr[E_i \mid E] = \frac{\Pr[E \cap E_i]}{\Pr[E]} = \frac{\Pr[E \mid E_i] \cdot \Pr[E_i]}{\sum_{j=1}^{k} \Pr[E \mid E_j] \cdot \Pr[E_j]}$$

Balls and Bins Questions

Setup: we have m balls and we want to put them in n bins.

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maximum load

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- What is "typically" the *maximum* number of balls in any bin?
- What is the *expected* number of bins with k balls in them?
- For what values of m do we expect to have *no empty bins*? (coupon collector)

Why Learn About Balls and Bins?

In next lectures, we are going to learn about and analyse *randomized algorithms*. While we will usually analyse the *expected running times* of the algorithms, we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

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Running time *small* with *high probability better than* small expected running time.

Strategy: devise randomized algorithm
with good expected running time
prove concentration of measure
around expectation

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Running time *small* with *high probability* *better than* small expected running time.

In **this lecture**, we will analyse random processes (*balls & bins*) which underlie several randomized algorithms!

Applications ranging from:

1. data structures
2. routing in parallel computers
3. many more!

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$B_{ij} = \begin{cases} 1 , & \text{if ball } i \to \text{bin } j \\ 0 , & \text{otherwise} \end{cases}$$

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all balls

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&= \sum_{i=1}^{m} \Pr\left[\text{ball } i \text{ in bin } j\right]
\end{aligned}$$

$$\mathbb{E}[B_{ij}] = 1 \cdot \Pr[\text{ball } i \text{ in bin } j] + 0 \cdot (1 - \ldots)$$

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When $m = n$, expectation of one ball per bin. How often will this actually happen?

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty. after m throws

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&= \sum_{i=1}^{n} \Pr[\text{bin } i \text{ is empty}]
\end{aligned}$$

$$\Pr[b \text{ does not fall in } i] = \left(1 - \frac{1}{n}\right)$$

$$\Pr[\text{bin } i \text{ empty}] = \left(1 - 1/n\right)^m$$

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&= \sum_{i=1}^{n} (1 - 1/n)^m \\
&= n \cdot (1 - 1/n)^m \approx n \cdot e^{-m/n}
\end{aligned}$$

$\left(1-\frac{1}{n}\right)^n \sim e^{-1}$

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When $m = n$, expected fraction of empty bins is $\frac{1}{e}$.

Head Scratching Moment

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Which expectation should I actually "expect"?

(example where expectation ≠ typical)

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

When $m = n$, second calculation had expectation of $1/e$ *fraction of empty bins*.

Which expectation should I actually "expect"?

As we mentioned earlier, this is where *concentration of probability measure* tries to address. It turns out that the *second random variable* (and thus second calculation) is concentrated around the mean (i.e., expectation).

So we "expect" (or it is "typical") to see around $1/e$-fraction of empty bins when $m = n$

Maximum load in a bin

What is the "typical" maximum number of balls in a bin?

As we saw in the previous slide, "typical" is related to concentration of probability measure.

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Let us first see a simpler problem, which is known as the *birthday paradox*: for what value of m do we expect to see two balls in one bin?

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

third ball

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdot \dots \cdot \left(1 - \frac{m-1}{n}\right) \leq e^{-1/n} \cdots e^{-\frac{m-1}{n}} \approx e^{\frac{-m^2}{2n}}$$

never any collision (only 1 ball)

second ball

k balls in bin

$1 - \frac{k}{n}$ $(k+1)^{th}$ ball doesn't cause collision

(avoid k occupied bins)

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

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This is $\leq 1/2$ when $m = \sqrt{2n \ln(2)}$. For $n = 365$, this is $m \approx 22.4$ for the probability that two people *(balls)* have birthday on the same date *(bins)* to become $\geq 1/2$.

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Thus, expect to see collision (two balls in the same bin) when $m = \Theta(\sqrt{n})$. This appears in several places:

- hashing
- factoring
- many more

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

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What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

union bound

$$\Pr[\text{bin 1 has } \geq k \text{ balls}] \leq \sum_{\substack{S \text{ subset}[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}]$$

$\cup$ (balls in $S \to$ bin 1), $S \subset [n]$, $|S| = k$

S ← set of k balls in bin 1

all possible sets of k balls

$$\Pr[S \to \text{bin } 1] = \prod_{i \in S} \Pr[\text{ball } i \to \text{bin } 1]$$

$$= \prod_{i \in S} \frac{1}{n} = \frac{1}{n^k}$$

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$$= \binom{n}{k} \cdot \frac{1}{n^k} \leq \left(\frac{ne}{k}\right)^k \cdot \frac{1}{n^k} = \frac{e^k}{k^k}$$

↑ # subsets of size k

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By union bound

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq \sum_{i=1}^{n} \Pr[\text{bin i has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k}$$

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

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$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

all bins have $\leq k$ load

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When will the above probability be large (say $>> 1/2$)?

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When $k \ln k > \ln n$. Setting $k = 3\frac{\ln n}{\ln \ln n}$ does it.

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This comes up in hashing and in analysis of approximation algorithms (for instance, best known approximation ratio for congestion minimization).

- Introduction
  - Probability basic notions
  - Balls and Bins
  - Analyses
- Coupon Collector and Power of Two Choices
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- Acknowledgements

Coupon Collector

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Why is this problem called the coupon collector problem?

Because we can formulate it in the following way:

- suppose each bin is a different coupon
- we buy one coupon at random (like kinder eggs/pack action cards)
- what is the number of coupons that we need to buy to collect all of them?

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Let X_i be the number of balls thrown to get from $i+1$ empty bins to i empty bins. Let X be the number of balls thrown until we have no empty bins.

$$X = \sum_{i=0}^{n-1} X_i$$

Coupon Collector

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What is $\mathbb{E}[X_i]$?

Coupon Collector

- $X_i \leftarrow \#$ balls thrown to get from i empty bins to $i-1$ empty bins
- $X \leftarrow \#$ balls thrown until we have no empty bins

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right]$$

What is $\mathbb{E}[X_i]$?

X_i geometric random variable with parameter $p = \frac{i}{n}$.

fraction of empty bins

Number of trials until the first success, where success probability p.

$$\Pr[X_i = k] = (1-p)^{k-1} \cdot p$$

k-th attemp succeed

fail in first k-1 attempts

Coupon Collector - Computing $\mathbb{E}[X]$

X_i takes values over $\mathbb{N}^*$

$$\mathbb{E}[X_i] = \sum_{k=1}^{\infty} k \cdot \Pr[X_i = k] = \sum_{k=1}^{\infty} \underbrace{\Pr[X_i \geq k]}_{(1-p)^{k-1}}$$

rearrange

$$\Pr[X_i \geq k] = \sum_{j=k}^{\infty} \Pr[X_i = j]$$

$$= \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-(1-p)} = \frac{1}{p} = \frac{n}{i}$$

$$\mathbb{E}[X_i] = 1 + \Pr[\text{first throw fails}] \cdot \mathbb{E}[X_i]$$

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{i} = n \cdot \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{H_n} \approx n \ln n$$

$$\Pr[X = k] = (1-p)^{k-1} \cdot p$$

$$1 \cdot p + \sum_{k=2}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

$$\sum_{j=1}^{\infty} (j+1) \cdot (1-p)^{j} \cdot p$$

$$= (1-p) \sum_{j=1}^{\infty} (j+1)(1-p)^{j-1} p$$

$$= (1-p) \sum_{j=1}^{\infty} (1-p)^{j-1} p + (1-p)\,\mathbb{E}[X]$$

$$\mathbb{E}[X] = \underbrace{1 \cdot p}_{\text{1st success}} + \underbrace{(1-p)}_{\text{1st fail}} \cdot \underbrace{\mathbb{E}[X+1]}_{\sum (k+1)\cdot(1-p)^k p}$$

$$= p + \underbrace{(1-p)\,\mathbb{E}[1]}_{1-p} + \underbrace{(1-p)}_{\text{pr. 1st throw fails}}\mathbb{E}[X]$$

$$= 1 + (1-p)\,\mathbb{E}[X].$$

Coupon Collector - Computing $\mathbb{E}[X]$

This $n \ln n$ bound shows up in:

- cover time of random walks in complete graph
- number of edges needed in graph sparsification
- many more places

Power of Two Choices

We now know that when n balls are thrown into n bins, the maximum load is $\Theta(\ln n / \ln \ln n)$ with constant probability.[1]

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Intuition/idea: let the height of a bin be the # balls in that bin. This process tells us that to get one bin with height $h + 1$ we must have at least two bins of height h.

We can bound # bins with height at least h (because this will tell us how likely it is to get to height $h + 1$).

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A bit more intuition

$N_h :=$ number of bins with height at least h

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- And only $n/256 = n/16^2 = n/2^{2^3}$ bins with height 6

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How do we turn this into a proof?
See [Mitzenmacher & Upfal, Chapter 14] and Lap Chi's notes.

Acknowledgement

- Lecture based largely on Lap Chi's notes and on [Motwani & Raghavan 2007, Chapter 3].
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L04.pdf`

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