Lecture 3: Concentration Inequalities

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

Why do we want concentration?

When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

"our algorithm will run in time $\approx T$ *very often*."

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- we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

expectation

typical event (what do you expect to see most of the time)

$$X = \begin{cases} 0 & \text{w.p. } 1/2 \\ 100 & \text{w.p. } 1/2 \end{cases}$$

$Y = i$ with prob. $1/100$ for $1 \leq i \leq 100$

$$\mathbb{E}[X] = 0 \cdot \frac{1}{2} + 100 \cdot \frac{1}{2} = 50$$

$$\mathbb{E}[Y] = \frac{1}{100} \sum_{i=1}^{100} i = \frac{1}{100} \cdot \frac{100 \cdot 101}{2} \approx 50$$

$$\Pr\left[X \in [40, 60] \right] = 0$$

$$\Pr[Y \in [40, 60]] \approx \frac{20}{100} = 0.2$$

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Often times in algorithm analysis, running time is *concentrated* around expectation. This *concentration of measure* proves that our algorithms will *typically* run in time close to expectation.

Today's inequalities

Theorem (Markov's Inequality)

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Theorem (Chebyshev's Inequality)

Let X be a discrete random variable. Then

$$\Pr[\underbrace{|X - \mathbb{E}[X]|}_{\text{X deviates from its expectation}} \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}, \quad \forall t > 0.$$

Today's inequalities II

sum of independent random variables

Theorem (Chernoff-Hoeffding's Inequality)

Let $X_1, \ldots, X_n$ be independent indicator variables such that $\Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Let $X = \sum_{i=1}^{n} X_i$ and $\delta > 0$. Then

$$\Pr[X \geq \underbrace{(1+\delta) \cdot \mathbb{E}[X]}_{\text{"small multiplicative" deviation}}] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mathbb{E}[X]},$$

exponential decay

and

$$\Pr[X \leq (1-\delta) \cdot \mathbb{E}[X]] \leq \exp\left(-\mathbb{E}[X] \cdot \delta^2/2\right).$$

Notation: $\exp(x) := e^x$

Markov's Inequality

$\{a_n\}_{n\geq 0}$ all a_i's distinct

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

Proof: $\mathbb{E}[X] = \sum_{n \geq 0} \Pr[X=n] \cdot n =$

$$= \underbrace{\sum_{n=0}^{t-1} \underbrace{\Pr[X=n]}_{\geq 0} \cdot n}_{\geq 0} + \underbrace{\sum_{n \geq t} \Pr[X=n] \cdot \underbrace{n}_{\geq t}}_{t \cdot \sum_{n\geq t} \Pr[X=n]}$$

$$\geq 0 + \underbrace{t \cdot \sum_{n \geq t} \Pr[X=n]}_{} = \underbrace{t \cdot \Pr[X \geq t]}_{}$$

$\boxtimes$

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Let X be a non-negative discrete random variable. Then

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- **Quicksort:** Expected running time of Quicksort is $2n \ln n$. Markov's inequality tells us that the runtime is at least $2cn \ln n$ with probability $\leq 1/c$, for any $c \geq 1$

$\mathbb{E}[T] = 2n \ln n$

$t = 2cn \ln n$

$\Pr[T \geq t] \leq \frac{1}{c}$

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- **Coin Flipping:** If we flip n fair coins, the expected number of heads is $n/2$. Markov's inequality tells us that $\Pr[\ \#\text{ heads } \geq 3n/4] \leq 2/3$

$$\Pr[\#\text{heads} \geq 3n/4] \leq \frac{n/2}{3n/4} = \frac{2}{3}$$

OBS: coin flipping is special (# heads = sum of independent random variables)

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Remark

Useful when we have no information beyond expected value (or when random variable difficult to analyze). Otherwise other inequalities much sharper!

Markov's Inequality

Some practice problems.

- Is Markov's inequality tight? Can you give an example?

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- Does it hold for general random variables (not just non-negative)?

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Some practice problems.

- Is Markov's inequality tight? Can you give an example?
- Does it hold for general random variables (not just non-negative)?
- Can it be modified to upper bound $\Pr[X \leq t]$?

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).

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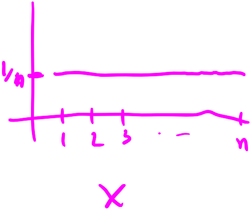
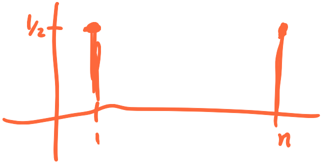
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$





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- Look at how far variable usually is from its expectation. How to do that?

How far X is from $\mathbb{E}[X]$: $|X - \mathbb{E}[X]|$

Can try to compute $\mathbb{E}\big[|X - \mathbb{E}[X]|\big]$

if close to expectation most of the time ← small
far from " " " " " ← large

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Let X be a discrete random variable. Then

$$\Pr[|X - \mathbb{E}[X]| \ge t] \le \frac{\mathrm{Var}[X]}{t^2}, \quad \forall t > 0.$$

Chebyshev's inequality

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- Its **Variance** is defined as $\mathrm{Var}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2]$
- and its **standard deviation** is $\sigma(X) := \sqrt{\mathrm{Var}[X]}$

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Let X be a discrete random variable. Then

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}[X]}{t^2}, \quad \forall t > 0.$$

Proof: we only know Markov. So let's use it:

$Y := (X - \mathbb{E}[X])^2$ $\quad Y \begin{cases} \text{discrete if } X \text{ discrete} \\ \geq 0 \end{cases}$

can use Markov!

By Markov: $\Pr[Y \geq t^2] \leq \frac{\mathbb{E}[Y]}{t^2} = \frac{\text{Var}[X]}{t^2}$

$= \Pr[|X - \mathbb{E}[X]| \geq t]$

Covariance

How do we measure the correlation between two random variables?

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Definition (Covariance)

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X, Y] := \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])].$$

We say that X, Y are *positively correlated* if $\mathrm{Cov}[X, Y] > 0$ and *negatively correlated* if $\mathrm{Cov}[X, Y] < 0$.

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Proposition

- $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]$
- *If X, Y are independent, then* $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

Practice problem : prove this!

Chebyshev & Covariance example

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- $X_i = \begin{cases} 1, & \text{if coin flipped heads} \\ 0, & \text{otherwise} \end{cases}$ i^{th} coin flip
- $X = \sum_{i=1}^{n} X_i$, and we know that X_i, X_j are independent

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- $X = \sum_{i=1}^{n} X_i$, and we know that X_i, X_j are independent
- By proposition:

(second bullet)

$$\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = n/4$$

$$\mathbb{E}[X_i] = 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 1/2$$

$$\mathrm{Var}[X_i] = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2] = \frac{1}{2}\left(1-\frac{1}{2}\right)^2 + \frac{1}{2}\left(0-1/2\right)^2$$

$$= 1/4$$

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- Chebyshev:

expectation of X

$$\underline{\Pr[X \geq 3n/4]} \leq \underline{\Pr[|X - n/2| \geq n/4]} \leq \frac{n/4}{(n/4)^2} = 4/n$$

bigger event space

Chebyshev

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Practice problem: Can you generalize Chebyshev's inequality to k^{th} order moments?

Sums of Independent Random Variables

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Law of large numbers: average of *independent, identically distributed variables* is *approximately* the *expectation* of the random variables. That is, if each X_i is an independent copy of random variable X

$$\frac{1}{n} \cdot \sum_{i=1}^{n} X_i \approx \mathbb{E}[X]$$

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independent identically distributed

Given n i.i.d. variables, how far are we from the expected value? And how often will we be close?
We want *non-asymptotic bounds*.

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for *any* value of n, when $X = X_1 + \cdots + X_n$.[1]

independent but not nec.

[1] Also works for sums of random variables which are not identically distributed!

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Simple Setting: we have n coin flips, each is head with probability p. So

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } X = \sum_{i=1}^{n} X_i.$$

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- Expected # heads: $n \cdot p$
- To bound upper tail, need to compute:

subsets of i elements

$$\Pr[X \geq k] \leq \sum_{i \geq k} \binom{n}{i} p^i (1-p)^{n-i}$$

turned 1 turn 0

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$$\Pr[X \geq k] \leq \underline{\sum_{i \geq k} \binom{n}{i} p^i (1-p)^{n-i}}$$

- Not easy to work with, hard to generalize

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Chernoff Bounds

Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } \underline{t > 0}.$$

exponential function is increasing function

Markov

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What do we gain by doing this?

- The *moment generating function*

$$M_X(t) := \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{i \geq 0} \frac{t^i}{i!} \cdot X^i\right] = \sum_{i \geq 0} \frac{t^i}{i!} \cdot \underbrace{\mathbb{E}\left[X^i\right]}_{i^{th} \text{ moment!}}$$

contains information about all moments!

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  contains information about all moments!

- If $X = X_1 + X_2$, where X_1, X_2 are independent, note that

$$\mathbb{E}[e^{tX}] \underset{\text{by def.}}{=} \mathbb{E}[e^{tX_1} e^{tX_2}] \underset{X_1, X_2 \text{ independent}}{=} \mathbb{E}[e^{tX_1}] \cdot \mathbb{E}[e^{tX_2}]$$

Chernoff Bounds for Bounded Variables

Example (Heterogeneous Coin Flips)

X_i's independent

Let $X_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{otherwise} \end{cases}$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$

1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$

$$\mu = \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \left(1 \cdot p_i + 0 \cdot (1-p_i)\right) = \sum_{i=1}^{n} p_i$$

Proof: $\Pr[X \geq (1+\delta)\mu] = \Pr[e^{tX} \geq e^{t(1+\delta)\mu}] \leq \frac{\mathbb{E}[e^{tX}]}{e^{t(1+\delta)\mu}}$

$$= \frac{1}{e^{t(1+\delta)\mu}} \cdot \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] = \frac{1}{e^{t(1+\delta)\mu}} \cdot \prod_{i=1}^{n} \Big(\underbrace{e^t \cdot p_i + 1 \cdot (1-p_i)}_{1 + p_i(e^t - 1)} \Big)$$

X_i's independent

$$1 + p_i(e^t-1) \le e^{p_i(e^t-1)}$$

$1 + x \le e^x \quad x \ge 0$

$$\le \frac{1}{e^{t(1+\delta)\mu}} \cdot \prod_{i=1}^{n} e^{p_i(e^t-1)} = \frac{1}{e^{t(1+\delta)\mu}} \exp\Big((e^t - 1) \cdot \sum_{i=1}^{n} p_i \Big)$$

$$= \left(\frac{e^{e^t-1}}{e^{t(1+\delta)}} \right)^{\mu} = \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right)^{\mu}$$

$t = \ln(1+\delta)$

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1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$
2. for $0 < \delta < 1$, $\Pr[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3}$

note that $0 < \delta < 1 \Rightarrow \frac{e^{\delta}}{(1+\delta)^{1+\delta}} \leq e^{-\delta^2/3}$

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2. for $0 < \delta < 1$, $\Pr[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3}$
3. for $R \geq 6\mu$, $\Pr[X \geq R] \leq 2^{-R}$

$R \geq 6\mu \iff \delta \geq 5$ in 1

Chernoff Bounds for Bounded Variables

What about the lower tail?

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Theorem (Heterogeneous Coin Flips - lower tail)

1. $\Pr[X \leq (1-\delta)\cdot\mu] \leq \left[\dfrac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu}$
2. *if* $0 < \delta < 1$ *then* $\Pr[X \leq (1-\delta)\cdot\mu] \leq e^{-\mu\delta^2/2}$

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Hoeffding's generalization

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Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^n X_i$. Then

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Proof uses *Hoeffding's lemma:* $\mathbb{E}[e^{t(X_i - \mathbb{E}[X_i])}] \leq \exp\left(\frac{t^2(b_i - a_i)^2}{8}\right)$

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected # heads is $n/2$. Chernoff-Hoeffding implies:

$$\Pr[|\#\text{ heads } - \mu| \geq \delta\mu] \leq 2\exp(-\mu\delta^2/3) = 2\exp(-n\delta^2/6)$$

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- From previous slides:

$$\text{Markov: } \Pr[\#\text{ heads } \geq 3n/4] \leq 2/3$$

$$\text{Chebyshev: } \Pr[\#\text{ heads } \geq 3n/4] \leq 4/n.$$

$$\text{Chernoff: } \Pr[\#\text{ heads } \geq 3n/4] \leq e^{-n/24}.$$

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- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)
- Chernoff-Hoeffding bounds also hold for *negatively correlated* variables, because all we need is

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- For instance: two edges appear in a random spanning tree is a negatively correlated event, thus Chernoff bounds are useful to analyze random spanning trees.

Acknowledgement

- Lecture based largely on Lap Chi's notes and [Motwani & Raghavan 2007, Chapters 3 and 4].
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L02.pdf`

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Mitzenmacher, Michael, and Eli Upfal (2017)
Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis.
Cambridge university press, 2017.