Lecture 1: Amortized Analysis

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

- Introduction
  - Why amortized analysis?
  - Types of amortized analyses
- Examples of Data Structures Using Amortized Analysis
  - Aggregate Analysis
  - Accounting Method
  - Potential Method
- Acknowledgements

Why Amortized Analysis?

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Worst or average-case complexity of data structures

Data Structure	search	insertion	deletion
Doubly-Linked List	$O(n)$	$O(1)$	$O(n)$
Ordered Array	$O(\log n)$	$O(n)$	$O(n)$
Hash Tables[a]	$O(1)$	$O(1)$	$O(1)$
Balanced Binary Search Trees[b]	$O(\log n)$	$O(\log n)$	$O(\log n)$

[a] Average-case, although worst-case search time is $\Theta(n)$

[b] Also average-case. Worst-case complexity is $O(\mathit{height})$ of the tree, which can be $\Theta(n)$.

Why Amortized Analysis?

In **amortized analysis**, one averages the *total time* required to perform a sequence of data-structure operations over *all operations performed*.

Upshot of amortized analysis: worst-case cost *per query* may be high for one particular query, so long as overall average cost per query is small in the end!

Remark

Amortized analysis is a *worst-case* analysis. That is, it measures the average performance of each operation in the worst case.

Types of amortized analyses

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3. **Potential Method:** one comes up with *potential energy* of a data structure, which maps each state of entire data-structure to a real number (its "potential"). Differs from accounting method because we assign credit to the data structure as a whole, instead of assigning credit to each operation.

One simple problem - several analyses

- **Input:** A binary counter C initially set to zero
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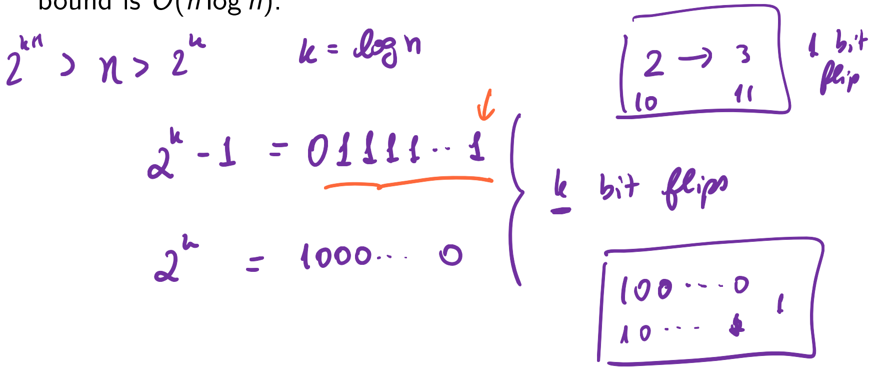

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- How many times will we "flip" the k^{th} bit?

$s = \lfloor \log n \rfloor$

0 1 1 · · · 1

k

$n/2^k$ times

need 2^k increments to change k^{th} bit

(very infrequently!)

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- How many times will we "flip" the k^{th} bit?
- Putting it all together, we get:

bit flips = $$\sum_{k=0}^{\lceil \log n \rceil} \lfloor n/2^k \rfloor < \sum_{k \geq 0} n/2^k = 2n$$

↳ # times we flipped k^{th} bit

Accounting method

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- That is, the *total charged* up to step ℓ is greater than or equal to the *actual cost* of all operations up to that point
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- If we manage to do the above, then

$$\text{Total cost} \leq \text{Total charged}$$

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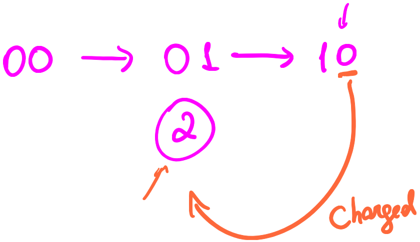

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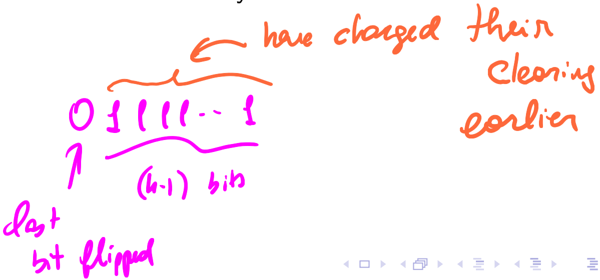

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- So, instead of *paying* for k bit flips in this increment, we *charge* at most 2: 0111..11 → 1000..00
  - one for setting a bit to 1. *actual cost*
  - and the other is the charge to "clear this bit" *clearing charge*

↳ happen in the future

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  - one for setting a bit to 1, *actual cost*
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- Total cost $\leq$ Total Charged $= 2 \times n$ ← # ops.

↳ charge per op

Example of the accounting method

$n = 5$ or $n = 9$

Formal Analysis of the accounting method

Charge to op i : $\boxed{\gamma_i = 2}$

1 for flipping bit

1 for clearing of this bit later

previous slide proved that "charging scheme" is valid

$$\sum_{i=1}^{\ell} \gamma_i \geq \sum_{i=1}^{\ell} c_i$$

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} 2 = 2n.$$

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$$\gamma_i = c_i + \Phi_i - \Phi_{i-1}$$

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- That is, total amortized cost is the *actual cost* of the operation plus the *change in potential* $(\Phi_1 - \Phi_0) + (\Phi_2 - \Phi_1) + (\Phi_3 - \Phi_2)$
- We have:

change in potential

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} (c_i + \Phi_i - \Phi_{i-1}) = \Phi_n - \Phi_0 + \sum_{i=1}^{n} c_i$$

total amortized cost

sum telescopes

actual cost

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- So if $\Phi_k - \Phi_0 \geq 0$ for all $k \geq 0$ (*valid potential function*) the total amortized cost is an *upper bound* on total cost.

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data structure

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- Potential:

$$\Phi_i = \text{ number of bits with value 1 at step } i$$

$\phi_0 = 0$

$\phi_k \geq 1 \quad k \neq 0$ (because k has at least 1 bit set to 1)

$\therefore$ potential is valid

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total # bits that we flipped

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increase pot. decrease pot.

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- Amortized cost:

$$\gamma_i = c_i + \Phi_i - \Phi_{i-1} = 2 \times (\#\text{ bits } 0 \to 1) = 2 \times 1 = 2$$

- Since each increment *only changes 1 bit from 0 to 1* each amortized cost is 2. $011\cdots1 \mapsto 100\cdots0$

Example of the potential method

Discussion of the potential method

$$\left(\sum c_i\right) + \underbrace{\Phi_n - \Phi_0}_{\geq 0} = \sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} 2 = 2n$$

$$\vee\!/$$

$$\sum c_i = \text{Total cost}$$

Acknowledgements

- Lecture largely based on Jeff Erickson's notes (with exercises!)

  `http://jeffe.cs.illinois.edu/teaching/algorithms/notes/09-amortize.pdf`

- More exercises and another example using all methods can also be found at the [CLRS] book, chapter 17. (see useful resources page)