Problem 1

1. Show that if a Markov chain with transition matrix P is irreducible and has a state i such that $P_{i,i} > 0$, then it is also aperiodic.

2. Let a, b be positive integers and consider the Markov chain with state space

$$\{(i, j) \mid 0 \leq i \leq a-1, \quad 0 \leq j \leq b-1\}$$

where i, j are integers, and the following transition mechanism: if the chain is in state (i, j) at time t, then at time $t+1$ it moves to $((i+1) \bmod a, j)$ with probability $1/2$ or to $(i, (j+1) \bmod b)$ with probability $1/2$.

Show that this Markov chain is irreducible, and show that it is aperiodic if, and only if, $\gcd(a, b) = 1$.

3. Consider a chessboard with a lone white king making random (king) moves, meaning that at each move, he picks one of the possible squares to move to, uniformly at random. Is the corresponding Markov chain irreducible and/or aperiodic? If so, what is the stationary distribution?

4. Same question as part 3, except now we have a lone knight in the chessboard.

Problem 2

Compute the mixing time (both upper bounds and lower bounds) of the dumbbell graph: the graph on $2n$ nodes that consists of two complete graphs on n nodes joined by a single edge.

Hint: try breaking up the graph into 4 pieces which converge to the stationary distribution in the appropriate time, and then show that any other probability distribution converges quickly to one of these pieces.

Problem 3

Suppose someone searches a keyword (like "car") and we would like to identify the webpages that are the most relevant for this keyword and the webpages that are the most reliable sources for this keyword (a page is a reliable source if it points to many of the most relevant pages).

First we identify the pages with this keyword and ignore all other pages. Then we run the following ranking algorithm on the remaining pages.

- Each vertex corresponds to a remaining page, and there is a directed edge from page i to page j if there is a link from page i to page j. Call this directed graph $G(V, E)$, where $n = |V|$.
- For each vertex i, we have two values $s(i)$ and $r(i)$, where intendedly $r(i)$ represents how relevant is this page and $s(i)$ represents how reliable it is as a source (the larger the values the better).
- We start from some arbitrary initial values, say $s(i) = 1/n$ for all i, as we have no idea of their relevance at the beginning.
- At each step, we update s and r (where s and r are vectors whose i^{th} entries are $s(i)$ and $r(i)$) as follows:
- First we update
$$r(i) = \sum_{j:(j,i)\in E} s(j)$$
for all i, as a page is more relevant if it is linked by many reliable sources.
- Then we update
$$s(i) = \sum_{j:(i,j)\in E} r(j)$$
for all i (using the just updated values $r(j)$), as a page is a more reliable source if it points to many relevant pages.
- To keep the values small, we let $R = \sum_{i=1}^{n} r(i)$ and $S = \sum_{i=1}^{n} s(i)$, and divide each $s(i)$ by S and each $r(i)$ by R.
- We repeat this step for many times to refine the values.

Let $s, r \in \mathbb{R}^n$ be the vectors of the s and r values.

1. Give a matrix formulation for computing s and r.
2. Suppose G is weakly connected (that is, when we ignore the direction of the edges, the underlying undirected graph is connected) and there is a self-loop at each vertex. Prove that there is a unique limiting s and a unique limiting r for any initial s as long as $s \geq 0$ and $s \neq 0$.

Hint: Perron-Frobenius

Problem 4

Let $S = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ and $T = \{x \in \mathbb{R}^n \mid Bx \leq c\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $B \in \mathbb{R}^{t \times n}$ and $c \in \mathbb{R}^t$. Given A, B, b, c as inputs, give a polynomial time algorithm for the problem of checking whether $S \subset T$.

Hint: Linear Programming

Problem 5

Suppose we are given a sequence of n linear inequalities of the form $a_i x + b_i y \leq c_i$, where x, y are variables and $a_i, b_i, c_i \in \mathbb{R}$. Collectively, these n inequalities describe a convex polygon P in the plane.

1. Describe a linear program whose solution describes the largest axis-aligned square that lies entirely inside P
2. Describe a linear program whose solution describes the maximum-perimeter axis-aligned rectangle that lies entirely inside P
3. Describe a linear program whose solution describes the largest circle that lies entirely inside P

Problem 6

Given a finite set of halfspaces $H_i := \{x \in \mathbb{R}^d \mid a_i^T x \leq b_i,\ a_i \in \mathbb{R}^d,\ b_i \in \mathbb{R}\}$, where $1 \leq i \leq n$ such that any set of $d+1$ halfspaces have a non-empty intersection, prove that there is a point lying in the intersection of all these halfspaces.