Lecture 9: Dimension Reduction

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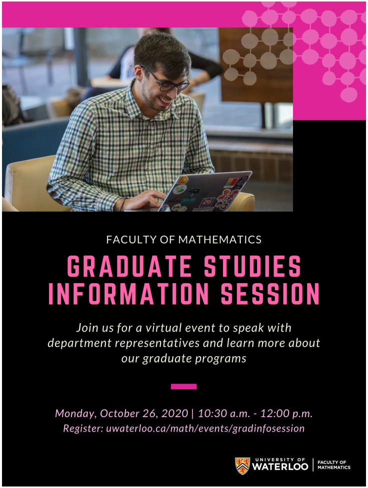
Overview

- Introduction
  - Administrivia
  - Why Reduce Dimensions?
  - Background: Continuous Probability Distributions
- Main Problem
  - Johnson-Lindenstrauss Lemma
- Acknowledgements

Grad School at UW!

Link to register:
https://uwaterloo.ca/math/events/gradinfosession



Why do we want low-dimensional objects?

When dealing with high-dimensional data, often times want to reduce dimension so that our algorithms run faster

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- Nearest Neighbor Search
- Large Scale Regression Problems
- Minimum Enclosing Ball
- Numerical linear algebra on large matrices
- Clustering

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- Cannot compress simplex while preserving all distances.

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Definition (Probability Density Function)

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- $$\Pr[a \leq X \leq b] = \int_a^b f(x)dx$$

Gaussian Random Variables (Normal Random Variables)

Definition

A real-valued random variable X has the *normal distribution* with

- mean μ
- variance σ^2,

denoted $X \sim \mathcal{N}(\mu, \sigma^2)$, if the probability density function of X, denoted $f_X : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is:

$$f_X(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2\right)$$

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Remark

When $\mu = 0$ and $\sigma = 1$ we say that X has *standard normal distribution*.

Properties of Gaussians

Proposition (Sums of Gaussians)

If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ *and* $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ *are independent Gaussians, then*

$$X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

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Proposition (General Linear Combinations)

If $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ *are independent random Gaussians, then*

$$\sum_{i=1}^{n} \alpha_i \cdot X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \alpha_i \cdot \mu_i \ \ , \ \ \sum_{i=1}^{n} (\alpha_i \cdot \sigma_i)^2\right).$$

χ^2 Random Variables

Definition

A real-valued random variable X has the *χ^2 distribution* with *k degrees of freedom*, denoted $X \sim \chi^2(k)$, if

$$X = Z_1^2 + \ldots + Z_k^2$$

where each $Z_i \sim \mathcal{N}(0, 1)$ is an independent standard normal random variable.

Concentration of χ^2 random variables

Lemma (Chernoff for $\chi^2(k)$)

If $Y = \sum_{i=1}^{k} X_i^2$ is a $\chi^2(k)$ random variable with k degrees of freedom (recall $X_i \sim \mathcal{N}(0,1)$), then

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- $$\Pr[Y > (1+\varepsilon)^2 \cdot k] = \Pr\left[e^{tY} > e^{t\cdot(1+\varepsilon)^2 \cdot k}\right] \leq \frac{\mathbb{E}[e^{tY}]}{e^{t\cdot(1+\varepsilon)^2\cdot k}}$$

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- By independence:

$$\mathbb{E}[e^{tY}] = \mathbb{E}\left[\exp\left(\sum_{i=1}^{k} t \cdot X_i^2\right)\right] = \prod_{i=1}^{k} \mathbb{E}[e^{tX_i^2}]$$

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value of $e^{tx_i^2}$

$$\mathbb{E}[e^{tX_i^2}] = \int_{-\infty}^{\infty} f_{X_i}(x) \cdot e^{tx^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \cdot e^{tx^2} dx$$

"probability that $x_i = x$"

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-(1-2t)x^2/2\right) dx$$

$-z^2/2$ $\quad \frac{dz}{\sqrt{1-2t}}$

$z = \sqrt{1-2t}\, x$

$\hookrightarrow \Rightarrow dz = \sqrt{1-2t}\, dx$

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- Change of variables $z = x\sqrt{1-2t}$

$$\mathbb{E}[e^{tX_i^2}] = \frac{1}{\sqrt{2\pi} \cdot \sqrt{1-2t}} \cdot \int_{-\infty}^{\infty} e^{-z^2/2} dz = \frac{1}{\sqrt{1-2t}}$$

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- Setting $t = (1/2)\cdot\left(1 - \frac{1}{(1+\varepsilon)^2}\right)$ above

$1 - 2t = (1+\epsilon)^{-2}$

$-t(1+\epsilon)^2 = \frac{1}{2}\left(1-(1+\epsilon)^2\right)$

$(1-2t)^{-k/2} = (1+\epsilon)^k$

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- Similar result for $\Pr[Y < (1-\varepsilon)^2 \cdot k]$ - **Practice problem.**

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- Main Problem
  - Johnson-Lindenstrauss Lemma
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Theorem (Johnson-Lindenstrauss Theorem)

Let $x_1, \ldots, x_m \in \mathbb{R}^n$ and $\varepsilon \in (0,1)$. For $d = O(\log(m)/\varepsilon^2)$ there exist points $y_1, \ldots, y_m \in \mathbb{R}^d$ such that:

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Moreover, the points $y_j = Lx_j$, where $L \in \mathbb{R}^{d \times n}$ is a matrix whose entries $L_{a,b} \sim \mathcal{N}(0,1)$, satisfies the above with probability $\geq 1 - 2/m$.

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- If one of the points is 0 then approximate norm of vectors as well!
- Independent of the original dimension n

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$$f(v) = (r_1^T v, r_2^T v, \ldots, r_d^T v)$$

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- By lemma, for *any* $u \in \mathbb{R}^n$, we have
  $\Pr[(1-\varepsilon)\cdot\|u\|_2 \leq \|L(u)\|_2 \leq (1+\varepsilon)\cdot\|u\|_2] \geq 1 - 2/m^3$

thus probability of failure (i.e. large distortion) is $\leq 2/m^3$

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- Apply this result and union bound to all vectors $x_a - x_b$.
- Probability any failure on the norm $\leq m^2 \cdot 2/m^3 = 2/m$.

#distances

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(even though in analysis we can flip the randomness, in the algorithm we would need to use GS to get random subspace)

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- So why not do that?
- A bit cumbersome to get random subspace (need to make L orthonormal - so need to use Gram-Schmidt)
- Just taking Gaussians do the trick without Gram-Schmidt!
- More convenient algorithmically

Proof of Johnson-Lindenstrauss Lemma

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Proof of upper tail: $\Pr[\|f(v)\|_2 > \sqrt{d} \cdot (1+\varepsilon)] < 1/m^3$

Proof of Johnson-Lindenstrauss Lemma

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Let $v \in \mathbb{R}^n$ such that $\|v\|_2 = 1$, $\varepsilon \in (0,1)$ and $d = O(\log(m)/\varepsilon^2)$. Let $r_1, \ldots, r_d \in \mathbb{R}^n$ be such that $r_i \sim \mathcal{N}(0,1)$. If we let $f : \mathbb{R}^n \to \mathbb{R}^d$ s.t.

$$f(v) = (r_1^T v, r_2^T v, \ldots, r_d^T v)$$

Then

$$\Pr\left[(1-\varepsilon) \ \leq \ \frac{\|f(v)\|_2}{\sqrt{d}} \leq \ (1+\varepsilon)\right] \geq 1 - 2/m^3.$$

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- Chernoff:

$$\Pr[\|f(v)\|_2^2 > d \cdot (1+\varepsilon)^2] < \exp(-(3/4) \cdot d\varepsilon^2) < 1/m^3$$

What if I don't like Gaussians?

- Can we even sample from a Gaussian?
- Same results also hold if pick a random matrix with entries uniformly from $\{-1, 1\}$ (Rademacher random variables).
- Proof a little more involved (see Jelani's notes for a proof)

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- Answer is *NO* in general.

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Can I also compress other norms?

- Answer is *NO* in general.
- [Brinkman, Charikar 2005]: For the ℓ_1-norm, where $\|x\|_1 = \sum_{i=1}^n |x_i|$, if want distortion $(1 + \varepsilon)$ dimension must be $\Omega(n^{1/(1+\varepsilon)^2})$

Acknowledgement

- Lecture based largely on Jelani Nelson's and Nick Harvey's notes.
- See Jelani's notes at
  http://web.mit.edu/minilek/www/jl_notes.pdf
- See Nick's notes at
  http://www.cs.ubc.ca/~nickhar/W12/Lecture6Notes.pdf

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