Lecture 8: Graph Sparsification

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Overview

- Introduction
  - Why Sparsify?
  - Warm-up Problem
- Main Problem
  - Graph Sparsification
- Acknowledgements

Why do we sparsify?

Often times graph algorithms for graphs $G(\overset{\text{vertices}}{V}, \overset{\text{edges}}{E})$ have runtimes which depend on $|E|$. If the graph is dense, i.e. $|E| = \omega(n^{1+c})$ then this may be *too slow*.

$|V| = n \qquad |E| = m$

Want something that is nearly-linear

$$O(n \cdot \underbrace{\text{polylog}(n)}_{\log^{\beta}(n)}) \qquad \beta > 0$$

Approximate answers

Why do we sparsify?

To sparsify is to make $|\tilde{E}|$ nearly linear

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- Used as primitives in many other algorithms (for instance, max-flow, sparsest cut, etc.)

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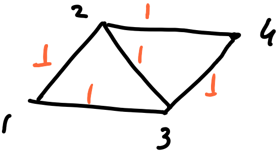
- Used as primitives in many other algorithms (for instance, max-flow, sparsest cut, etc.)
- Applications in network connectivity

Graph Cuts

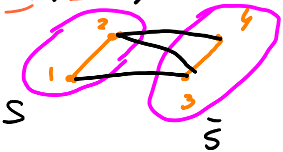
Definition (Graph Cut)

If $G(\underline{V}, \underline{E}, \underline{w})$ is a weighted graph, a *cut* is a partition of the vertices into two non-empty sets $V = S \sqcup \underset{:= V\setminus S}{\overline{S}}$. The *value* of a cut is the quantity

$$\underline{w(S,\overline{S})} := \sum_{e\in E(S,\overline{S})} w_e.$$



$(\{1,2\},\{3,4\})$



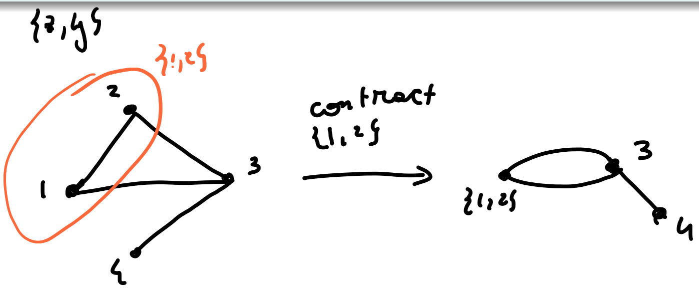
$w(S,\bar{S}) = 3 = |E(S,\bar{S})|$

Contraction of Edges

Definition (Edge Contraction)

Let $G(V, E)$ be a graph. If $e = \{u, v\} \in E$ is an edge of G, then the *contraction* of e is a new graph $H(V \cup \underline{\{z\}} \setminus \underline{\{u, v\}}, \underline{F})$ where we replace the vertices u, v by *one* vertex z, and any edge $\{u, x\} =: f \in E \setminus \{e\}$ is replaced by $\{z, x\} \in F$.

$\{v, y\}$



Randomized Minimum Cut

- **Input:** undirected unweighted graph $G(V, E)$
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    - Pick uniformly random edge and contract it

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- **Output:** minimum cut $(S, \overline{S})$, with high probability
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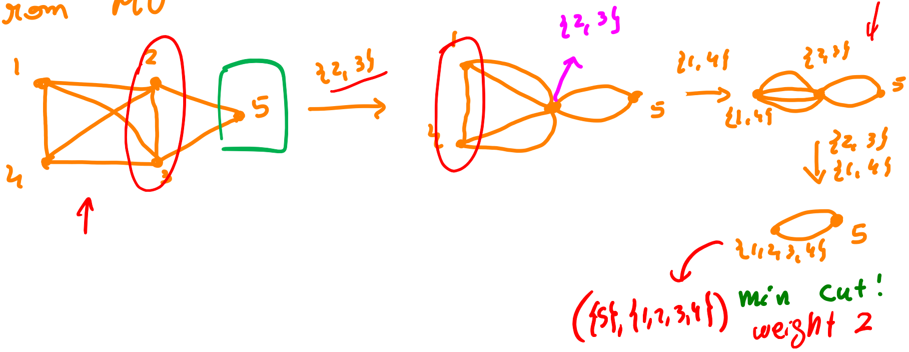
- Output the two subsets encoded by the two remaining vertices.

From MU



({5}, {1, 2, 3, 4}) min cut!
weight 2

Analysis

Why does this work?

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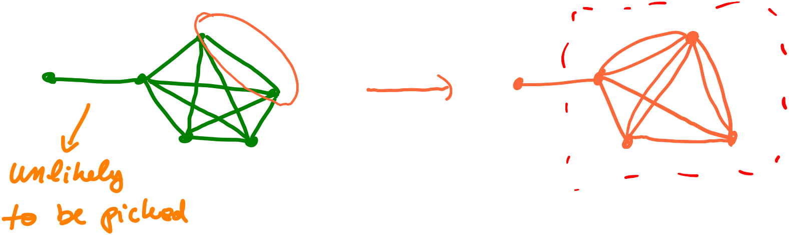
Intuition: picking a random edge uniformly at random “favours” *small cuts* (i.e. preserves them) with higher probability.



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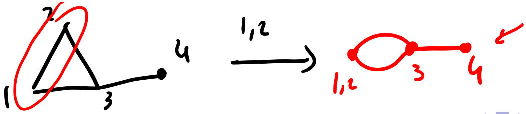
Remark

The value of the minimum cut only increases or stays the same after contraction.

After contracting 1 and 2, only cuts that remain are cuts $(S, \bar{S})$ where $\underline{1,2} \in S$ (and these have same value).



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$= \omega(S, \bar{S})$

- Let $(S, \overline{S})$ be a minimum cut, and $k := |E(S, \overline{S})|$. If we never contract an edge from $\underline{E(S, \overline{S})}$, the algorithm succeeds.

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- Probability that an edge from $E(S, \overline{S})$ is contracted in the i^{th} iteration (conditioned on cut still alive)
- Each vertex is a cut, so each vertex has degree $\geq \underbrace{k}_{\text{size of min-cut}} \Rightarrow$

$(\{v\}, V-\{v\})$

$$\geq \frac{(n-i+1)\cdot k}{2} \quad \text{edges remain.}$$

vertices in graph after i-1 iterations

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- Contracting random edge, probability we kill cut $(S, \overline{S})$ is

$$= \underbrace{|E(S, \overline{S})|}_{\text{\#edges in my cut}} \cdot \frac{1}{(\#\text{ edges})} \leq k \cdot \frac{2}{(n-i+1) \cdot k} = \boxed{\frac{2}{n-i+1}}$$

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$1 - \frac{2}{n-i+1}$

survive

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- $\Pr[(S, \overline{S}) \text{ survives}] \geq \underline{(1 - 2/n)} \cdot \underline{(1 - \overset{2/n-1}{\cancel{3/n}})} \cdots (1 - 2/3) = \underline{2/n(n-1)}$

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- If we repeat for t times, failure probability is

$$\leq \left(\underbrace{1 - \frac{2}{n(n-1)}}_{\text{Probability that my iteration fails}}\right)^t$$

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$$\leq \left(1 - \frac{2}{n(n-1)}\right)^t$$

- setting $t = 2n(n-1)$ then

$$\leq \left(1 - \frac{2}{n(n-1)}\right)^t \leq \exp\left(-\frac{2t}{n(n-1)}\right) = e^{-4} < \frac{1}{3}$$

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- For running time improvements, see [Motwani & Raghavan 2007, Chapter 10.2]

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This is all good, but we haven't “sparsified” anything so far!

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Graph Sparsification

Definition (Weight of a cut)

Let $G(V, E, w)$ be undirected weighted graph. For any cut $(S, \overline{S})$, let the weight of $(S, \overline{S})$ be

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w(e).$$

If all weights are 1 then graph said to be "unweighted"

$$w(S, \bar{S}) = |E(S, \bar{S})|$$

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Definition (Sparse Graph)

We say that a graph $G(V, E)$ is *sparse* if $|E| = \tilde{O}(|V|)$.

$\tilde{O}(n^\alpha)$ means any function $f: \mathbb{N} \to \mathbb{N}$ s.t. $f(n) = O(n^\alpha \cdot \log^\beta n)$ for some $\beta > 0$ (just hiding the log factors)

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Question

How to make a graph sparse (nearly linear # edges) while approximating the value of every cut of a graph?

Graph Sparsification

- **Input:** graph $G(V, E, w_G)$, $\varepsilon > 0$.

$$n = |V|, \ m = |E|.$$

- **Output:** graph $H(V, F, w_H)$ such that *for every cut* $(S, \overline{S})$, we have

$$(1-\varepsilon) \cdot w_G(S, \overline{S}) \leq w_H(S, \overline{S}) \leq (1+\varepsilon) \cdot w_G(S, \overline{S})$$

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Algorithm:

- Let $p \in (0, 1)$ be a parameter.
- For each edge $e \in E(G)$, with probability p, make e an edge of H with weight $w_H(e) = 1/p$.

Graph Sparsification

Idea:

- Set p to get correct expected value for both $\#$ edges in H and the value of each cut $(S, \overline{S})$ in H.

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Theorem ([Karger, 1993])

Let c be the value of the min-cut of G. Set

$$p = \frac{15 \ln n}{\varepsilon^2 \cdot c}.$$

Graph H given by algorithm from previous slide **approximates all cuts of** *G and has $O(p \cdot |E|)$ edges with probability $\geq 1 - 4/n$.*

Graph Sparsification

- Take a cut $(S, \overline{S})$. Suppose $k := w_G(S, \overline{S})$. Let
  $$X_e = \begin{cases} 1, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$

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  $$X_e = \begin{cases} 1, & \text{if edge } e \textbf{ included in } H \\ 0, & \text{otherwise} \end{cases}$$
- $$\mathbb{E}[|F|] = \sum_{e \in E} \mathbb{E}[X_e] = \sum_{e \in E} \underbrace{(p \cdot 1 + (1-p) \cdot 0)}_{\mathbb{E}[X_e]} = p \cdot |E|$$

linearity expectation

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- $$\mathbb{E}[w_H(S, \overline{S})] \overset{\text{linearity expectation}}{=} \sum_{e \in E(S, \overline{S})} \mathbb{E}[w_H(e)] = \sum_{e \in E(S, \overline{S})} \overbrace{\left(p \cdot \frac{1}{p} + (1-p) \cdot 0\right)}^{\mathbb{E}[w_H(e)]}$$
$$= |E(S, \overline{S})| \underset{\text{G is unweighted (i.e. } w_G(e) \in \{0,1\})}{=} k = \underbrace{w_G(S, \overline{S})}_{\text{definition}}$$

Graph Sparsification - Concentration

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- $w_H(S, \overline{S})$ is a sum of independent random variables w_e
- Chernoff Bound: (why Chernoff if vars not in $\{0,1\}$?)

$$\Pr[|w_H(S, \overline{S}) - k| \geq \varepsilon \cdot k] \leq 2 \exp\left(-\frac{\varepsilon^2 kp}{3}\right) = 2n^{-5k/c}$$

$$w_e = \frac{1}{p} \cdot X_e \qquad X_S = \sum_{e \in E(S,\bar{S})} X_e = p \cdot \sum w_e = w_H(S,\bar{S}) \cdot p$$

$$\Pr[|w_H(S,\bar{S}) - k| \geq \varepsilon k] = \Pr[|X_S - pk| \geq \varepsilon pk]$$

$$\underset{\substack{\downarrow \\ \text{Chernoff} \\ \text{on } X}}{\leq} 2e^{-\left(\frac{\varepsilon^2 \cdot pk}{3}\right)}$$

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- **Observation:** probability that large cut violated is *much smaller*, and there are *not many small cuts*!
- So we can do a clever union bound!

Number of Cuts Lemma

Lemma (Number of small cuts)

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Practice problem: generalize our earlier proof on the # minimum cuts to this case.

Union Bound on # Cuts

Pr[some cut is violated] $\leq$

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$$\Pr[\text{some cut is violated}] \leq$$

$$\leq \sum_{S \subseteq V} \Pr[(S, \overline{S}) \text{ is violated}]$$

union bound

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$$\le \underbrace{\sum_{\alpha = 1,2,4,8,\dots}}_{\alpha = 2^i,\ i \in \{0,1,\dots,\log n\}} \underbrace{\sum_{\substack{S \subseteq V \\ \alpha c \le |w_G(S,\overline{S})| \le 2 \cdot \alpha c}}}_{\text{grouping cuts based on their weights}} \Pr[(S, \overline{S}) \text{ is violated}]$$

breaking up sum

$$2^i \cdot c \le w_G(S, \bar{S}) \le 2^{i+1} c$$

Remember: G unweighted, so

$$w_G(S, \bar{S}) = |E(S, \bar{S})|$$

Union Bound on # Cuts

$$\Pr[\text{some cut is violated}] \leq$$

$$\leq \sum_{S \subseteq V} \Pr[(S, \overline{S}) \text{ is violated}]$$

$$\leq \sum_{\alpha=1,2,4,8,\ldots} \sum_{\substack{S \subseteq V \\ \alpha c \leq |w_G(S,\overline{S})| \leq 2\cdot\alpha c}} \Pr[(S, \overline{S}) \text{ is violated}]$$

$$\leq \sum_{\alpha=1,2,4,8,\ldots} \underbrace{n^{4\alpha}}_{\downarrow} \cdot \Pr[(S, \overline{S}) \text{ is violated} \mid \alpha c \leq |w_G(S, \overline{S})| \leq 2 \cdot \alpha c]$$

upper bd. # cuts with weight between αc and 2αc (by our lemma)

(i.e. # edges in cut)

Union Bound on # Cuts

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$$\leq \sum_{S \subseteq V} \Pr[(S, \overline{S}) \text{ is violated}]$$

$$\leq \sum_{\alpha=1,2,4,8,\ldots} \quad \sum_{\substack{S \subseteq V \\ \alpha c \leq |w_G(S,\overline{S})| \leq 2 \cdot \alpha c}} \quad \Pr[(S, \overline{S}) \text{ is violated}]$$

$$\leq \sum_{\alpha=1,2,4,8,\ldots} n^{4\alpha} \cdot \Pr[(S, \overline{S}) \text{ is violated} \mid \alpha c \leq |w_G(S, \overline{S})| \leq 2 \cdot \alpha c]$$

$$\leq \sum_{\alpha=1,2,4,8,\ldots} n^{4\alpha} \underbrace{\cdot\, 2n^{-5\alpha c/c}}_{\text{our Chernoff bound}}$$

Union Bound on # Cuts

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$$= \sum_{\alpha = 1,2,4,8,\dots} n^{-\alpha} \leq 4/n$$

↳ sum dominated by first term.

Union Bound on # Cuts

$$
\begin{aligned}
&\Pr[\text{some cut is violated}] \leq \\
&\leq \sum_{S \subseteq V} \Pr[(S,\overline{S}) \text{ is violated}] \\
&\leq \sum_{\alpha=1,2,4,8,\ldots} \sum_{\substack{S \subseteq V \\ \alpha c \leq |w_G(S,\overline{S})| \leq 2\cdot\alpha c}} \Pr[(S,\overline{S}) \text{ is violated}] \\
&\leq \sum_{\alpha=1,2,4,8,\ldots} n^{4\alpha} \cdot \Pr[(S,\overline{S}) \text{ is violated} \mid \alpha c \leq |w_G(S,\overline{S})| \leq 2\cdot\alpha c] \\
&\leq \sum_{\alpha=1,2,4,8,\ldots} n^{4\alpha} \cdot 2n^{-5\alpha c/c} \\
&= \sum_{\alpha=1,2,4,8,\ldots} n^{-\alpha} \leq 4/n
\end{aligned}
$$

Another application of Chernoff gives us that H has the right number of edges $|F| \approx p \cdot |E|$ (i.e., sparse)

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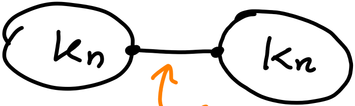
Counterexample :



(dumbell graph)

cannot remove this edge!

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- **[Benczur, Karger 1996]:** without minimum cut assumption, just sample non-uniformly in clever way!

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- Sample edge e with probability $p_e = \Theta\left(\frac{\log n}{\varepsilon^2 \cdot s_e}\right)$ and weight $1/p_e$.

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's Lecture 1 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L01.pdf`
- See Lap Chi's Lecture 3 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L03.pdf`
- See Mohsen's notes for the general Benczur-Karger algorithm `https://people.inf.ethz.ch/gmohsen/AA18/Notes/S1.pdf`.

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