Lecture 4: Hashing

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Overview

- Introduction
  - Hash Functions
  - Why is hashing?
  - How to hash?
- Succinctness of Hash Functions
  - Coping with randomness
  - Universal Hashing
  - Hashing using 2-universal families
  - Perfect Hashing
- Acknowledgements

Computational Model

Before we talk about hash functions, we need to state our model of computation:

Definition (Word RAM model)

In the word RAM[a] model:

- all elements are integers that fit in a machine word of w bits
- Basic operations (comparison, arithmetic, bitwise) on such words take $\Theta(1)$ time
- We can also access *any* position in the array in $\Theta(1)$ time

[a] RAM stands for Random Access Model

What is hashing?

We want to store n elements (keys) from the set $U = \{0, 1, \ldots, m-1\}$, where $m >> n$, in a data structure that supports *insertions, deletions, search* "as efficiently as possible."

$m >> n$ and $\log m < w$

(each element from $\{0, 1, \ldots, m-1\}$ fits in one word of memory)

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- Insertion: $O(1)$, Deletion: $O(1)$, Search: $O(1)$
- Memory: $O(m \log(m))$ *(this is very bad!)*

>> our entire memory!

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Want to also achieve optimal memory $\boxed{O(n \log(m))}$. For this we will use a technique called *hashing*.

- A *hash function* is a function $h : U \to [0, n-1]$, where $|U| = m >> n$.
- A *hash table* is a data structure that consists of:
  - a table T with n cells $[0, n-1]$, each cell storing $O(\log(m))$ bits
  - a hash function $h : U \to [0, n-1]$

From now on, we will define memory as *# of cells*. "$w = O(\log m)$"

Why is hashing useful?

- Designing efficient data structures (dictionaries) for searching
- Data streaming algorithms
- Derandomization
- Cryptography
- Complexity Theory
- many more

Challenges in Hashing

Setup:

- Universe $U = \{0, \ldots, m-1\}$ of size $m >> n$ where n is the size of the range of our hash function $h : U \rightarrow [0, n-1]$
- Store $O(n)$ elements of U (keys) in hash table T (which has n cells)

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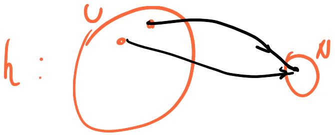
Ideally, want hash function to map *different keys* into *different locations*.

Definition (Collision)

We say that a *collision* happens for hash function h with inputs $x, y \in U$ if $x \neq y$ and $h(x) = h(y)$.

By pigeonhole principle, impossible to achieve without knowing keys in advance.



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Will settle for: # collisions *small with high probability*.

Our solution: family of hash functions

Construct *family* of hash functions $\mathcal{H}$ such that the *number of collisions* is **small** with **high probability**, when we pick hash function uniformly at random from the family $\mathcal{H}$.

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Simplest version to keep in mind:

$$\Pr_{h \in_R \mathcal{H}} [h(x) = h(y)] \leq \frac{1}{\text{poly}(n)} \quad \boxed{\forall x \neq y \in U}$$

collision

small

when take random function h from $\mathcal{H}$

For any pair $x \neq y \in U$ $\Pr_{\text{random } h}$ (x and y collide) is small

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Simplest version to keep in mind:

$$\Pr_{h \in_R \mathcal{H}}[h(x) = h(y)] \le \frac{1}{\mathrm{poly}(n)} \qquad \forall x \ne y \in U$$

Assumptions:

- keys are independent from hash function we choose.
- we **do not** know keys in advance (even if we did, nontrivial problem!)

Question

Still could have collisions. How do we handle them?

Random Hash Functions?

Natural to consider following approach:

From all functions $h : U \to \underbrace{[0, n-1]}_{N}$, just pick one uniformly at random.

Random Hash Functions?

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balls bins

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{0, 1, ..., m-1}

This setting is same as our balls-and-bins setting!

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So, if we have to store n keys:

- Expected number of keys in a location: 1 (expected # balls in a bin)
- maximum number of collisions (max load) in one particular location: $O(\log n / \log \log n)$ keys

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Could also pick *two* random hash functions and use *power of two choices*.
Collision bound becomes $\boxed{O(\log\log n)}$. $h_1(x)$ $h_2(x)$

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- Storing entire function $h : U \to [0, n-1]$ require $\tilde{O}(m \log n)$ bits (way too much space!) for each $x \in U$ $(x, h(x)) \to \log n$
- Even if we only stored the elements we saw, would require $\boxed{O(n)}$ time to evaluate $h(x)$ (need to decide if we had already computed it!)

$x_0, x_1, \ldots, x_{n-1}, x_n$

$x_{n-1} = x_0$

$(x_0, h(x_0))$

$(x_1, h(x_1))$

$h(x_{n-1}) = h(x_0)$

$\Theta(m)$ cells

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Remark

Thus, for random function all operations (insert, delete, search) take $O(n \log m)$ time (at best!)

in RAM $O(n)$ time!

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Remark

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How do we cope with the computational problem that arose with randomness?

How to cope with "hardness" of randomness?

We want something that is *random-like* (few collisions w.h.p.) but *easy to compute/represent*.

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linear time

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poly(m) functions, as each function takes at most $\boxed{O(\log m) \text{ bits}}$ to describe. Thus these are *succinct functions* (easy to describe and compute) which have *random-like* properties!

$2^{O(\log m)}$

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Part of *derandomization/pseudorandomness*: huge subfield in TCS!

k-wise independence

Weaker notion of independence.

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Definition (Full Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be (fully) independent if they satisfy

$$\Pr\left[\bigcap_{i=1}^{n} X_i = a_i\right] = \prod_{i=1}^{n} \Pr[X_i = a_i]$$

$X_1 = a_1, X_2 = a_2 \ldots X_n = a_n$

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Definition (k-wise Independence)

A set of random variables $X_1, \ldots, X_n$ are said to be k-wise independent if for any set $\boxed{J \subset [n]}$ such that $\underline{|J| \le k}$ they satisfy

only small sets variables behave as if independent

$$\Pr\left[\bigcap_{i \in J} X_i = a_i\right] = \prod_{i \in J} \Pr[X_i = a_i]$$

Pairwise independence

When $k = 2$, k-wise independence is called *pairwise independence*.

Example (XOR pairwise independence)

Given b uniformly random bits $Y_1, \ldots, Y_b$, we can generate $2^b - 1$ pairwise independent random variables as follows:

$$X_S := \bigoplus_{i \in S} Y_i \qquad \boxed{S \subseteq [b] \cancel{\setminus \emptyset} \quad S \neq \emptyset}$$

$a \oplus b$

a	b	$a \oplus b$
0	0	0
1	0	1
0	1	1
1	1	0

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- Why are they even random?

$$X_{\{1,2,3\}} = \underbrace{Y_1 \oplus Y_2}_{1} \oplus \underbrace{Y_3}_{\substack{1 \to 0 \\ 0 \to 1}}$$

$$X_S = \begin{cases} 1 & 1/2 \\ 0 & 1/2 \end{cases}$$

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- Why are they pairwise independent?

$S_1 \neq S_2$ there is variable

$Y_1 \in S_1 \setminus S_2$

S_1, S_2 subsets of $[b]$

$$\Pr[X_{S_1} = a_1 \text{ and } X_{S_2} = a_2] = \frac{1}{4}$$

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- Why are they pairwise independent?
- Are they also 3-wise independent?

No $X_{\{1\}}$ $X_{\{2\}}$ $X_{\{1,2\}}$ not 3-wise ind.

Pairwise independence II

integers mod p.

Example (Pairwise independence in $\mathbb{F}_p$)

Let p be a prime number. Given 2 uniformly random variables $Y_1, Y_2 \sim \underbrace{[0, \ldots, p-1]}_{\mathbb{F}_p}$, generate p pairwise independent random variables as follows:

$$X_i := Y_1 + i \cdot Y_2 \mod p \qquad i \in [0, p-1]$$

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$$X_0 = Y_1$$

$$X_1 = Y_1 + Y_2 \quad \rightsquigarrow \quad \Pr[X_i = a] = \frac{1}{p}$$

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- Why are they pairwise independent?

X_i , X_j
$\shortparallel \quad \shortparallel$
$a \quad b$

$Y_1 + i Y_2 = a$
$Y_1 + j Y_2 = b$

1 solution

$$\boxed{\Pr[X_i = a, X_j = b] = \frac{1}{p^2}} \qquad \det\begin{pmatrix} 1 & i \\ 1 & j \end{pmatrix} = j - i \text{ (invertible)}$$

$\shortparallel$
$\Pr[X_i = a] \cdot \Pr[X_j = b]$

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NO. Practice problem.

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- Why are they even random?
- Why are they pairwise independent?
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Can think of these random variables as picking a random line over a finite field. If we only know one point of the line, the second point is still uniformly random. However two points determine the line.

Universal Hash Functions

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Definition (Universal Hash Functions)

Let U be a universe with $|U| \geq n$. A family of hash functions $\mathcal{H} = \{h : U \to [0, n-1]\}$ is *k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$, we have

$$\Pr_{h \in_R \mathcal{H}} [h(u_1) = h(u_2) = \ldots = h(u_k)] \leq 1/n^{k-1}$$

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Definition (Strongly Universal Hash Functions)

$\mathcal{H} = \{h : U \to [0, n-1]\}$ is *strongly k-universal* if, *for any* **distinct** elements $u_1, \ldots, u_k \in U$ and *for any* values $y_1, \ldots, y_k \in [0, n-1]$, we have

$$\Pr_{h \in_R \mathcal{H}} [\underline{h(u_1) = y_1}, \ldots, \underline{h(u_k) = y_k}] \leq 1/n^k$$

random like.

Relation to k-wise independent random variables

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Relation to k-wise independent random variables

What do the previous definitions have to do with random variables?

Family $\mathcal{H}$ is *strongly k-universal* if the random variables $h(0), \ldots, h(|U| - 1)$ are *k-wise independent*.

Can use random variables to construct universal hash functions!

Strongly 2-universal families of hash functions

Let p be a prime number, $U = [0, p-1]$.

Proposition

$$\mathcal{H} = \{h_{a,b}(\underbrace{x}_{\in U}) := a \cdot x + b \mod p \mid a, b \in [0, p-1]\}$$

is strongly 2-universal.

Proof xc $a = Y_1$ $b = Y_2$ random vars proof

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Proposition

Let $U = [0, p^k - 1] \equiv \underbrace{[0, p-1]^k \setminus \{(0, \ldots, 0)\}}_{\mathbb{F}_p^k \setminus \vec{0}}$ *and* $\mathbf{a} = (a_0, \ldots, a_{k-1}) \in U$

$$\mathcal{H} = \{h_{\mathbf{a},b}(\mathbf{x}) := \underbrace{\mathbf{a} \cdot \mathbf{x}}_{a_0 \cdot x_0 + \cdots + a_k \cdot x_k} + b \mod p \quad | \quad \mathbf{a} \in U, b \in [0, p-1]\}$$

$h : \mathbb{F}_p^k \to \mathbb{F}_p$

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2-universal families of hash functions

What if my hast table size is not a prime?

Proposition

$$\mathcal{H} = \{h_{a,b}(x) := (a \cdot x + b \mod p) \mod n \quad | \quad a, b \in [0, p-1]\}$$

is 2-universal (but not strongly 2-universal).

Practice problem: prove the proposition above.

k-universal families of hash functions

Can we construct k-universal families of hash functions like this?

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- YES! Instead of constructing random lines (degree 1 polynomials), can construct random univariate polynomials of degree $k-1$

$ax+b$

$$\boxed{a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0}$$

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- Random degree $k-1$ polynomials are k-wise independent!
- Practice problem: prove this!

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Remark

- In XOR example, our function takes $O(n)$ storage space, and $O(n)$ time to compute.[a]
- In $\mathbb{F}_p$ examples, our function takes $O(1)$ storage space and $O(1)$ time to compute![b]

$h_{a,b}$ store a, b $O(1)$

$ax+b$ $O(1)$ time

[a] Reminder that we assume that $n < 2^w$.
[b] We assume that $p < 2^w$.

Theorem from Probability

Theorem (Markov's inequality)

If X is a non-negative random variable and $t > 0$, we have:

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$$

Hashing with 2-universal families

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- Only need to choose $a, b \in [0, p-1]$ to store a function from $\mathcal{H}$.
- Computation time of $h_{a,b}$ is also $O(\log m)$

Hashing with 2-universal families

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- $\mathcal{H} = \{h_{a,b}(x) := (a \cdot x + b \mod p) \mod n \quad | \quad a, b \in [0, p-1]\}$
- Only need to choose $a, b \in [0, p-1]$ to store a function from $\mathcal{H}$.
- Computation time of $h_{a,b}$ is also $O(\log m)$
- Can this hash function match chain hashing parameters?
  ($O(\log \log n)$ search time)

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Lemma (Maximum number of collisions)

The expected number of collisions using a 2-universal hash family is

$$\ell^2/2n$$

ℓ = # elements that will hash

Hashing with 2-universal families

ℓ = # elements from U that we will hash

Lemma (Maximum number of collisions)

The expected number of collisions using a 2-universal hash family is

$$\leq \ell^2/2n$$

$$X_{ij} = \begin{cases} 1 & \text{if keys } i \text{ and } j \text{ are mapped to same location} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{1 \leq i < j \leq \ell} X_{ij} \quad \text{\# collision pairs}$$

$$E[X] = \sum_{i,j} E[X_{ij}] = \sum_{i,j} \frac{1}{n} \leq \frac{\ell^2}{2n}$$

2-universal

By Markov we have that

$$\Pr\left[X > \underbrace{\ell^2/n}_{\text{many collisions}}\right] < \frac{1}{2}$$

Hashing with 2-universal families

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Thus, by Markov's inequality, we have $\frac{t^2}{2} \sim \binom{t}{2} \leq \frac{\ell^2}{n}$ with prob. $> 1/2$

Lemma (Maximum load of entry of hash table)

With probability $\geq 1/2$ the ~~number of collisions~~ max load of cell using a 2-universal hash family is

$$\leq \sqrt{\frac{2\ell^2}{n}}.$$

$t \leq \sqrt{\frac{2\ell^2}{n}}$

When $\ell \approx n$ (as is usually assumed in hashing), we expect $\sqrt{2n}$. collisions

t entries in a cell $\Rightarrow$ have $\geq \binom{t}{2}$ collisions

Perfect Hashing

How to build a hash table with $O(1)$ search time and $O(n)$ memory? Can we still do it with a 2-universal family of hash functions?

Setup: we know the keys in advance (static data structure)

(in lecture: how can we use 2-universal hashing to get $O(1)$ search time but that required $O(n^2)$ memory)

Perfect Hashing

How to build a hash table with $O(1)$ search time and $O(n)$ memory? Can we still do it with a 2-universal family of hash functions?

Corollary

If $h \in \mathcal{H}$ is a random hash function from a 2-universal family of hash functions, then for any set $S \subseteq U$ of size $\ell \leq \sqrt{n}$, the probability of h being perfect for S is at least $1/2$.

no collisions on S

Proof: There is no collision with probability $\geq 1/2$.

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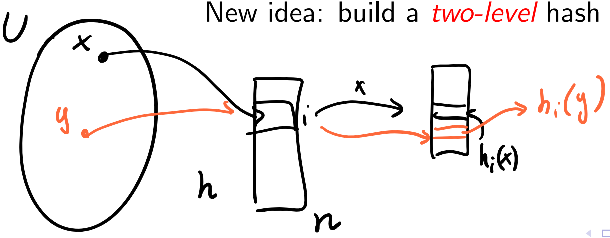
New idea: build a *two-level* hash table!



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Theorem

The two-level approach gives perfect hashing scheme.

how many hash functions have we used? ($O(n)$) ✓

memory is also $O(n)$

Proof (sketch) of Theorem

Remark we know keys in advance S set of keys

Approach: pick first layer hash function h uniformly at random. Test h on S.

With probability $\geq 1/2$ max # collisions in one bin is $\leq \sqrt{n}$. We will get good hash fcn h with constant # tries.

Assume that max # collisions (h, S) is $\leq \sqrt{n}$

$\ell_i \leftarrow$ load at i^{th} cell of hash table given by h.

know : $l_i \le \sqrt{n}$ (because h is good for S)

and $\sum_{i=0}^{n-1} l_i = n$ $(= |S|)$

Lemma if take h_i random hash function from $\boxed{h_i : S \to l_i^2}$ h_i is perfect for the l_i elements that map into i^{th} cell of first table h.

We showed that expected # collisions is 0.

Bound on Memory: $\sum_{i=0}^{n-1} l_i^2$ (bad because $\sqrt{n} \cdot (\sqrt{n})^2 = n^{3/2}$)

We also know (because h is good for S)

w.h.p. # collisions is $\leq \ell^2/n$

(expected # collisions is $\leq \ell^2/2n$ ← lemma)

when $\ell = n$ (our case) we get that

collisions $\leq n$

$$\sum_{i=0}^{n-1} \ell_i^2 = O(\text{\# collisions of } h)$$

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L05.pdf`

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