Lecture 20: Online Algorithms & k-server

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Overview

- Administrivia
- Online Algorithms: Randomized Lower Bounds
- k-server on a line
- Conclusion
- Acknowledgements

Rate this course!

Please log in to

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from *November 24th until December 7th* and provide us with your evaluation and feedback on the course!

- This would really help me figuring out what worked and what didn't for the course
- And whether I should put memes or gifs into my slides...
- Teaching this course is also a learning experience for me :)

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A deterministic online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

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→ expectation over random bits used by A

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- Main question: which entry of the cache to delete?
- Cost function: *number of cache misses*
- Simplification: assume we only have cache and main memory.

Lower Bound - Deterministic Paging Algorithms

Theorem

Any deterministic algorithm for paging with k pages is at least k-competitive!

- Proof by trolling.[1] Let's use $k + 1$ pages, and let A be our paging algorithm.

[1] Common lower bound technique for online algorithms, also commonly used online as well :)

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- When offline algorithm deletes a page, it's next delete happens after at least k steps.

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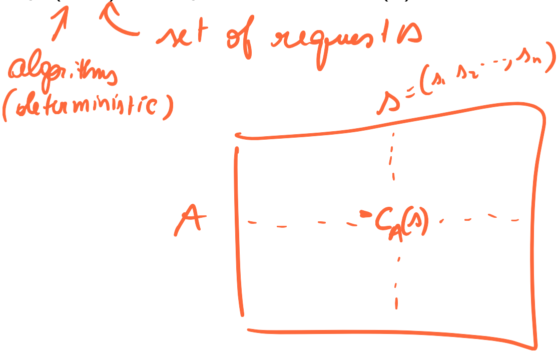

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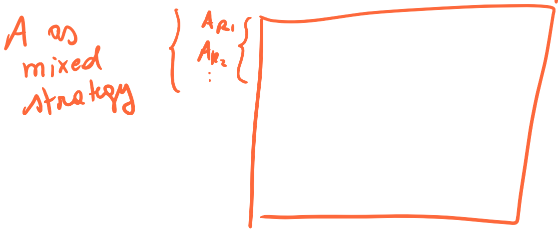

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Theorem (Yao's minimax principle)

If for some input distribution, no deterministic algorithm is k-competitive, then no randomized algorithm is k-competitive!

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$s = (s_1, s_2, \ldots, s_n)$

$s_i \in \{1, 2, \ldots, k, k+1\}$

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coupon collector

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$$\frac{n}{k} \le \log k \cdot \frac{n}{k \log n}$$

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Paging problem   cache size 2

3 different pages

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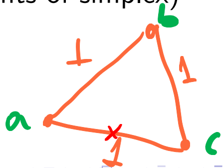
2-server problem (size of cache)

3 equidistant requests



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- Today's Simplification: assume X is a *line*. Think $X = \mathbb{R}$

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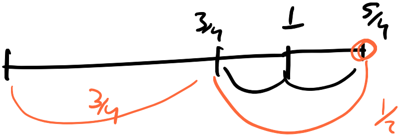

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$s_1 = \frac{3}{4} \quad s_2 = \frac{5}{4} \quad s_3 = \frac{3}{4} \quad s_4 = \frac{5}{4} \quad \cdots$

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6. Best strategy: put A on $3/4$, B on $5/4$

$\frac{3}{4}$ $\frac{5}{4}$

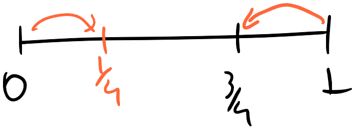
$\text{Cost}_{\text{OPT}}(s) = 1$

$\text{Cost}_{\text{Greedy}}(s) = \Omega(n)$

Attempt 2: Double Coverage (DC)

- If request falls between two servers, move both towards request at same rate until one reaches it

(simplification : never query exactly at half)



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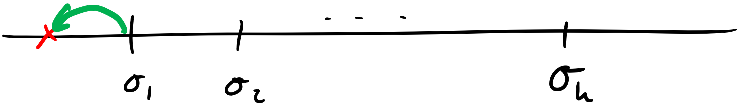

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Practice problem: prove this!

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5. Potential Function:
   - match each server from DC to a server of OPT
   - track changes as requests come

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- Total ammortized cost:

$$\sum_{t=1}^{n} \gamma_t = \sum_{t=1}^{n} c_t + \Phi_t - \Phi_{t-1}$$

$$= \Phi_n - \Phi_0 + \sum_{t=1}^{n} c_t$$

final initial actual total cost

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$$\gamma_t = c_t + \Phi_t - \Phi_{t-1}$$

- Total ammortized cost:

$$\sum_{t=1}^{n} \gamma_t = \sum_{t=1}^{n} c_t + \Phi_t - \Phi_{t-1}$$

$$= \Phi_n - \Phi_0 + \sum_{t=1}^{n} c_t$$

- If potential function is always *non-negative*

$$\sum_{t=1}^{n} c_t \leq \Phi_0 + \sum_{t=1}^{n} \gamma_t$$

DC Analysis - Potential Function

Main idea: have the *ammortized cost* per request be (a multiple of) the cost of OPT, while the actual cost is the cost of DC.

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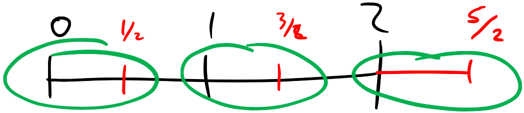

$S_0 = 1 + 2 + 1 = 4$

$M_0 = 3/2$

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- Note that $\Phi_t \geq 0$ at all times
- Use Amortized Analysis to compute amortized cost of DC
- Break requests into two parts:
  - First account for OPT move
  - Then account for DC move

DC Analysis - Potential Function

1. OPT moves

DC Analysis - Potential Function

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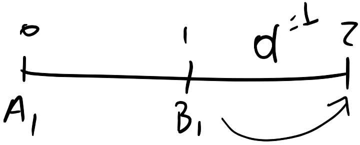
    - If OPT moves a distance d, the distance from the moved server to the matched DC's server increases by d



$$d(A_1^{(t)}, B_1^{(t)}) = 1$$

$$d(A_1^{(t+1)}, B_1^{(t+1)}) = 2$$

DC Analysis - Potential Function

1. OPT moves
   - If OPT moves a distance d, the distance from the moved server to the matched DC's server increases by d
   - So $M_{t+1} \leq M_t + d$

have matching DC OPT of cost $M_t + d$

$\Rightarrow M_{t+1} \leq M_t + d$

DC Analysis - Potential Function

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   - Thus potential increased (so far) by $\Phi_{t+1} - \Phi_t \leq k \cdot d$

k × distance that OPT travelled

$$\Phi_{t+1} - \Phi_t$$

$$k \cdot M_{t+1} + S_{t+1} - k \cdot M_t - S_t$$

$$k(M_{t+1} - M_t) \leq k \cdot d$$

$$M_{t+1} - M_t \leq d$$

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DC Analysis - Potential Function

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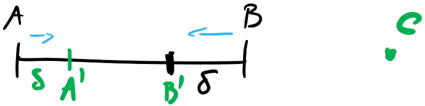
      - Both servers move a distance δ.

DC Analysis - Potential Function



$$d(A', C) + d(B', C) = d(A, C) - \delta + d(B, C) + \delta$$
$$= d(A, C) + d(B, C)$$

2. DC moves
   1. The request falls between two servers A and B. Say that B is taken to the location requested.
      - Both servers move a distance δ.
      - Thus pairwise distances decrease by 2δ (because they are in a line)

$$d(A', B') = d(A, B) - 2\delta$$

$$\boxed{S_{t+1} = S_t - 2\delta}$$

DC Analysis - Potential Function

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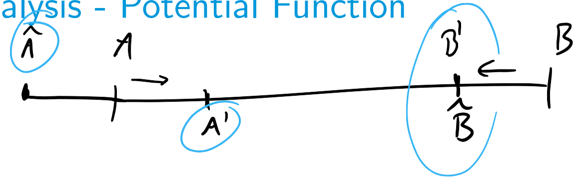
      - B has match at destination (problem constraint)

from OPT

DC Analysis - Potential Function



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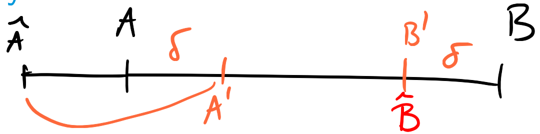
DC Analysis - Potential Function



if $(A, \hat{A})$ matched in M_t — $d(A', \hat{A}) = d(A, \hat{A}) + \delta$

$(B, \hat{B})$ matched in M_t — $d(B', \hat{B}) = d(B, \hat{B}) - \delta$

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have another matching of cost M_t

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DC Analysis - Potential Function

$$M_{t+1} - M_t \leq 0$$

$$\Phi_{t+1} - \Phi_t = k\underbrace{(M_{t+1} - M_t)}_{\leq 0} + \underbrace{(S_{t+1} - S_t)}_{= -2\delta}$$

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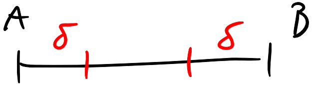
      - Potential Change: $\Phi_{t+1} - \Phi_t \leq k \cdot 0 - 2 \cdot \delta = -2 \cdot \delta$

DC Analysis - Potential Function



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$$\Delta\Phi \leq -k \cdot \delta + (k-1) \cdot \delta = -\delta$$

$k(M_{t+1} - M_t)$ $(S_{t+1} - S_t)$

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DC Analysis - Wrapping Up

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- By our potential function inequality, we have:

$$\sum_{t=1}^{n} c_t \leq \Phi_0 + \sum_{t=1}^{n} \gamma_t$$

C_A $\quad 0 \quad \leq k \cdot d_t(OPT)$

$\leq k \cdot C_{OPT}$

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- Since $\gamma_t \leq k \cdot d$ whenever OPT moves d, and $\gamma_t \leq 0$ when OPT doesn't move, we have that $\sum_t \gamma_t \leq k \cdot C_{opt}$

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- Since $\gamma_t \leq k \cdot d$ whenever OPT moves d, and $\gamma_t \leq 0$ when OPT doesn't move, we have that $\sum_t \gamma_t \leq k \cdot C_{opt}$
- Since Φ_0 is the initial state, we can regard it as constant (even 0, if require that servers start at a certain place)

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- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*
- Saw how to use *minimax theorem* in *Yao's principle* to prove lower bounds for randomized online algorithms.

Acknowledgement

- Lecture based largely on:
  - Lectures 18 & 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Karger's Lecture 18 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s23-onlineRandomLb.pdf`
- See Karger's Lecture 20 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf`

References I

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Randomized Algorithms