Lecture 18: Hardness of Approximation

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

November 16, 2020

Overview

- Background and Motivation
  - Why Hardness of Approximation?
  - How do we prove Hardness of Approximation?
  - Hardness of Approximation - Example
- Proofs & Hardness of Approximation
- Conclusion
- Acknowledgements

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others

John Nash

Gödel → Von Neumann

Johnson

Garey and others

in TCS or CO — combinatorial optimization problems

intractable problems

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?
    - design algorithm which is efficient on "most" instances and always gives us the exact/best answer

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?
  - design algorithm which is efficient on "most" instances and always gives us the exact/best answer
  - design (always) efficient algorithm, but finds sub-optimal solutions

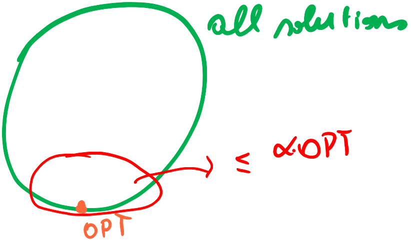
Approximation Algorithms

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?
  - design algorithm which is efficient on "most" instances and always gives us the exact/best answer
  - design (always) efficient algorithm, but finds sub-optimal solutions

    Approximation Algorithms

  - For $\alpha \geq 1$, an algorithm is *α-approximate* for a minimization (maximization) problem if on every input instance the algorithm finds a solution with cost $\leq \alpha \cdot OPT$ $(\geq \frac{1}{\alpha} \cdot OPT)$.



Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?
  - design algorithm which is efficient on "most" instances and always gives us the exact/best answer
  - design (always) efficient algorithm, but finds sub-optimal solutions

    Approximation Algorithms
  - For $\alpha \geq 1$, an algorithm is *α-approximate* for a minimization (maximization) problem if on every input instance the algorithm finds a solution with cost $\leq \alpha \cdot OPT$ ($\geq \frac{1}{\alpha} \cdot OPT$).
- For some problems, it is possible to prove that even the design of approximation algorithms for certain values of α is impossible, unless P $=$ NP (in which case we would have an exact algorithm).

Hardness of Approximation

α-approximation $\mathcal{P}$ $\Leftarrow$ exactly solve $\mathcal{P}$

Why Study Hardness of Approximation?

- Since the 50s and 60s (before we "formally knew" about NP) researchers from many areas noticed that certain combinatorial problems were much harder to solve than others
- What do we do when we see such a hard problem?
  - design algorithm which is efficient on "most" instances and always gives us the exact/best answer
  - design (always) efficient algorithm, but finds sub-optimal solutions

    Approximation Algorithms

  - For $\alpha \geq 1$, an algorithm is *α-approximate* for a minimization (maximization) problem if on every input instance the algorithm finds a solution with cost $\leq \alpha \cdot OPT$ ($\geq \frac{1}{\alpha} \cdot OPT$).
- For some problems, it is possible to prove that even the design of approximation algorithms for certain values of α is impossible, unless P $=$ NP (in which case we would have an exact algorithm).

  Hardness of Approximation

- Important to know the limits of efficient algorithms!

How do we Prove Hardness of Approximation?

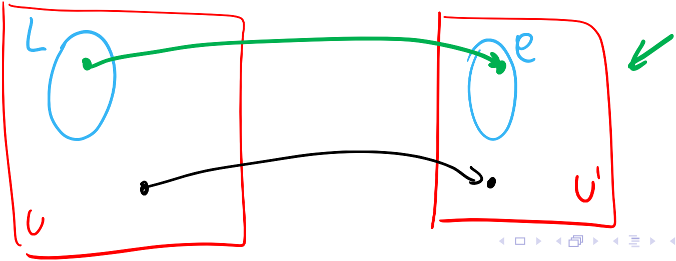
- When we prove that a combinatorial problem $\mathcal{C}$ is NP-hard, we usually pick our favorite NP-complete combinatorial problem L and we show a *reduction* that

How do we Prove Hardness of Approximation?

- When we prove that a combinatorial problem $\mathcal{C}$ is NP-hard, we usually pick our favorite NP-complete combinatorial problem L and we show a *reduction* that
  - maps every YES instance of L to a YES instance of $\mathcal{C}$
  - maps every NO instance of L to a NO instance of $\mathcal{C}$

So if we had an algorithm that solves $\mathcal{C}$ then this algorithm would also solve L!

(size of the reduction must be small)



How do we Prove Hardness of Approximation?

- When we prove that a combinatorial problem $\mathcal{C}$ is NP-hard, we usually pick our favorite NP-complete combinatorial problem L and we show a *reduction* that
  - maps every YES instance of L to a YES instance of $\mathcal{C}$
  - maps every NO instance of L to a NO instance of $\mathcal{C}$
- Let's do this for the CLIQUE problem. Input for CLIQUE is (G, k)
  - maps every YES instance of SAT to a YES instance of CLIQUE
  - maps every NO instance of SAT to a NO instance of CLIQUE

If φ is a boolean formula, then we would map φ to graph G_φ that had k-clique in case φ is satisfiable. $\varphi \mapsto (G_\varphi, k)$ YES

If φ not satisfiable, we could map it to (H_φ, k) where H_φ has cliques of size $\leq k-1$. NO

How do we Prove Hardness of Approximation?

- When we prove that a combinatorial problem $\mathcal{C}$ is NP-hard, we usually pick our favorite NP-complete combinatorial problem L and we show a *reduction* that
  - maps every YES instance of L to a YES instance of $\mathcal{C}$
  - maps every NO instance of L to a NO instance of $\mathcal{C}$
- Let's do this for the CLIQUE problem. Input for CLIQUE is (G, k)
  - maps every YES instance of SAT to a YES instance of CLIQUE
  - maps every NO instance of SAT to a NO instance of CLIQUE

How do we Prove Hardness of Approximation?

- When we prove that a combinatorial problem $\mathcal{C}$ is NP-hard, we usually pick our favorite NP-complete combinatorial problem L and we show a *reduction* that
  - maps every YES instance of L to a YES instance of $\mathcal{C}$
  - maps every NO instance of L to a NO instance of $\mathcal{C}$
- Let's do this for the CLIQUE problem. Input for CLIQUE is (G, k)
  - maps every YES instance of SAT to a YES instance of CLIQUE
  - maps every NO instance of SAT to a NO instance of CLIQUE
- For hardness of approximation what we would like is a (more robust) reduction of the form:

How do we Prove Hardness of Approximation?

$\varphi \longmapsto \begin{cases} (G_\varphi, k) & \text{YES of CLIQUE} \\ (H_\varphi, k) & \text{VERY MUCH NO} \end{cases}$ H_φ does not have $k/3$-cliques

MAX-CLIQUE: input graph G, output maximum clique of G

- Let's do this for the CLIQUE problem. Input for CLIQUE is (G, k)
  - maps every YES instance of SAT to a YES instance of CLIQUE
  - maps every NO instance of SAT to a NO instance of CLIQUE
- For hardness of approximation what we would like is a (more robust) reduction of the form:
  - maps every YES instance of SAT to a YES instance of CLIQUE
  - maps every NO instance of SAT to a VERY-MUCH-NO instance of CLIQUE

had 2-approximation alg. for MAX CLIQUE

MAX-CLIQUE(G_φ) $\longmapsto$ have a $k/2$ clique — YES

MAX-CLIQUE(H_φ) $\longmapsto$ have a $\leq k/3$ clique — NO

- Background and Motivation
  - Why Hardness of Approximation?
  - How do we prove Hardness of Approximation?
  - Hardness of Approximation - Example
- Proofs & Hardness of Approximation
- Conclusion
- Acknowledgements

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

- For any path $p_0 \to p_1 \to \cdots \to p_t$ in X, *length* of the path is sum of distances traveled

$$\sum_{i=0}^{t-1} d(p_i, p_{i+1})$$

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

- For any path $p_0 \to p_1 \to \cdots \to p_t$ in X, *length* of the path is sum of distances traveled

$$\sum_{i=0}^{t-1} d(p_i, p_{i+1})$$

- **Output:** find a cycle that reaches all points in X of shortest length.

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

- For any path $p_0 \to p_1 \to \cdots \to p_t$ in X, *length* of the path is sum of distances traveled

$$\sum_{i=0}^{t-1} d(p_i, p_{i+1})$$

- **Output:** find a cycle that reaches all points in X of shortest length.
- Definitely a problem we would like to solve
  - Efficient route planning (mail system, shuttle bus pick up and drop off...)

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

- For any path $p_0 \to p_1 \to \cdots \to p_t$ in X, *length* of the path is sum of distances traveled

$$\sum_{i=0}^{t-1} d(p_i, p_{i+1})$$

- **Output:** find a cycle that reaches all points in X of shortest length.
- Definitely a problem we would like to solve
  - Efficient route planning (mail system, shuttle bus pick up and drop off...)
- One of the famous NP-complete problems

Hardness of Approximation - TSP

1. *General* TSP *without* repetitions (General TSP-NR)

Hardness of Approximation - TSP

1. *General* TSP *without* repetitions (General TSP-NR)
   - if $P \neq NP$ then there is no poly-time constant-approximation algorithm for General TSP-NR.

Hardness of Approximation - TSP

1. *General* TSP *without* repetitions (General TSP-NR)
   - if $P \neq NP$ then there is no poly-time constant-approximation algorithm for General TSP-NR.
   - More generally, if there is any function $r : \mathbb{N} \rightarrow \mathbb{N}$ such that $r(n)$ computable in polynomial time, then it is hard to $r(n)$-approximate General TSP-NR if we assume that $P \neq NP$

Hardness of Approximation - TSP

1. *General* TSP *without* repetitions (General TSP-NR)
   - if $P \neq NP$ then there is no poly-time constant-approximation algorithm for General TSP-NR.
   - More generally, if there is any function $r : \mathbb{N} \to \mathbb{N}$ such that $r(n)$ computable in polynomial time, then it is hard to $r(n)$-approximate General TSP-NR if we assume that $P \neq NP$
2. How does one prove any such hardness of approximation?

By *reduction* ~~to~~ from another NP-hard problem.

Hardness of Approximation - TSP

1. *General* TSP *without* repetitions (General TSP-NR)
   - if $P \neq NP$ then there is no poly-time constant-approximation algorithm for General TSP-NR.
   - More generally, if there is any function $r : \mathbb{N} \to \mathbb{N}$ such that $r(n)$ computable in polynomial time, then it is hard to $r(n)$-approximate General TSP-NR if we assume that $P \neq NP$
2. How does one prove any such hardness of approximation?

   By *reduction* to another NP-hard problem.
3. In our case, let's reduce it to the *Hamiltonian Cycle Problem*

Theorem

If there is an algorithm M which solves TSP without repetitions with α-approximation, then $P = NP$.

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that

Hamiltonian cycle problem

TSP

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that
   - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that
   - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)
   - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \\ (1 + \alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \end{cases}$

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that
   - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)
   - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \\ (1 + \alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \end{cases}$
3. If G has a Hamiltonian Cycle, then OPT for the TSP is of value $\leq |V|$

$OPT(H) \leq |V|$

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that
   - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)
   - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \\ (1 + \alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \end{cases}$
3. If G has a Hamiltonian Cycle, then OPT for the TSP is of value $\leq |V|$
4. If G has no Hamiltonian Cycle, then OPT for TSP must use an edge not in V, thus value is $\geq (1 + \alpha) \cdot |V|$

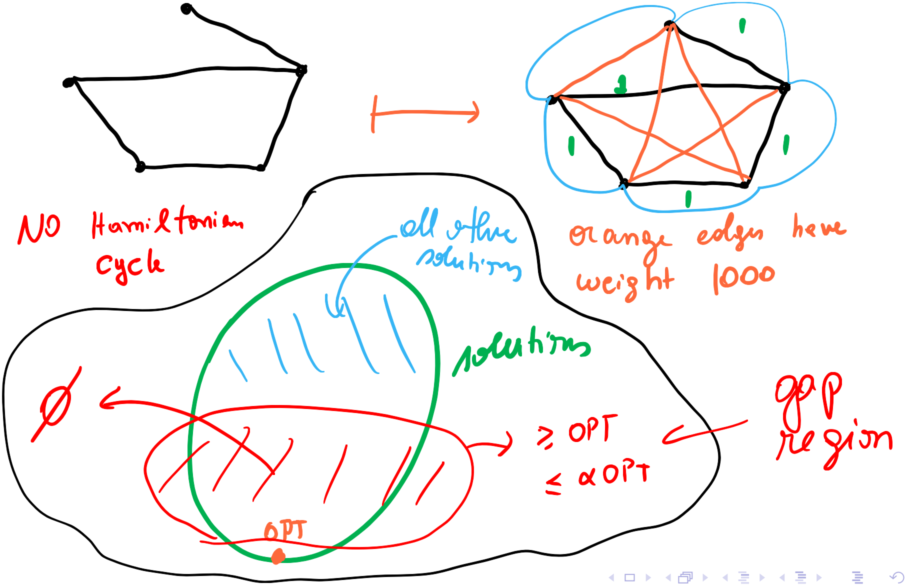
$OPT(H) \geq$ weight of $\{u,v\} \notin E = (1+\alpha) \cdot |V|$

Hardness of Approximation

1. **Hamiltonian Cycle Problem:** given a graph $G(V, E)$, decide whether there exists a cycle $\mathcal{C}$ which passes through every vertex at most once.
2. If we had an algorithm M which solved the α-approximate TSP without repetition problem, then
   - from graph $G(V, E)$, construct weighted graph $H(V, F, w)$ such that
   - All edges $\{u, v\} \in F$ (that is, H is the complete graph on V)
   - $w(u, v) = \begin{cases} 1, & \text{if } \{u, v\} \in E \\ (1+\alpha) \cdot |V|, & \text{if } \{u, v\} \notin E \end{cases}$
3. If G has a Hamiltonian Cycle, then OPT for the TSP is of value $\leq |V|$
4. If G has no Hamiltonian Cycle, then OPT for TSP must use an edge not in V, thus value is $\geq (1+\alpha) \cdot |V|$
5. Thus, M on input H will output a Hamiltonian Cycle of G, if G has one, or it will output a solution with value $\geq (1+\alpha) \cdot |V|$

inputs only have cycles of weight $\{ \leq |V| \leftarrow$; $\geq (1+\alpha)|V|$

Discussion of Proof



Complexity Classes

- **NP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine V, such that:

$$x \in L \Leftrightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)} \text{ s.t. } V(x, w) = 1$$

w ← witness that x in in the language

$$x \in L \Rightarrow \exists\ w \in \underbrace{\{0,1\}^{poly(|x|)}}_{small} \text{ s.t. } V(x,w) = 1$$

$$x \notin L \Rightarrow \forall\ w \in \{0,1\}^{poly(|x|)} \text{ s.t. } V(x,w) = 0$$

Complexity Classes

- **NP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine V, such that:

$$x \in L \Leftrightarrow \exists w \in \{0,1\}^{\mathrm{poly}(|x|)} \text{ s.t. } V(x,y) = 1$$

- **BPP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine M, such that:

$$x \in L \Leftrightarrow \Pr_{R \in \{0,1\}^{\mathrm{poly}(|x|)}} [M(x,R) = 1] \geq 2/3$$

$R \leftarrow$ random string used by the randomized algorithm

$\tilde{M} := M(\cdot\ ,$ random string$)$

Complexity Classes

- **NP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine V, such that:

$$x \in L \Leftrightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)} \text{ s.t. } V(x,y) = 1$$

- **BPP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine M, such that:

$$x \in L \Leftrightarrow \Pr_{R \in \{0,1\}^{\text{poly}(|x|)}}[M(x,R) = 1] \geq 2/3$$

- **RP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine M, such that:

$$\rightarrow \quad x \in L \Rightarrow \Pr_{R \in \{0,1\}^{\text{poly}(|x|)}}[M(x,R) = 1] \geq 2/3$$

$$\rightarrow \quad x \notin L \Rightarrow \Pr_{R \in \{0,1\}^{\text{poly}(|x|)}}[M(x,R) = 1] = 0$$

Randomized algorithms with one-sided error

Complexity Classes

- **NP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine V, such that:

$$x \in L \Leftrightarrow \exists w \in \{0,1\}^{\mathrm{poly}(|x|)} \text{ s.t. } V(x,y) = 1$$

- **BPP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine M, such that:

$$x \in L \Leftrightarrow \Pr_{R \in \{0,1\}^{\mathrm{poly}(|x|)}}[M(x,R) = 1] \geq 2/3$$

- **RP:** Set of languages $L \subseteq \{0,1\}^*$ such that there exists a poly-time Turing Machine M, such that:

$$x \in L \Rightarrow \Pr_{R \in \{0,1\}^{\mathrm{poly}(|x|)}}[M(x,R) = 1] \geq 2/3$$
$$x \notin L \Rightarrow \Pr_{R \in \{0,1\}^{\mathrm{poly}(|x|)}}[M(x,R) = 1] = 0$$

- **co-RP:** languages $L \subseteq \{0,1\}^*$ s.t. $\overline{L} \in RP$

Proof Systems

A proof system looks like this:

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
    - The prover must provide proofs in a certain format
    - The verifier can use algorithms from a certain complexity class for verification

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")
3. A prover writes down a proof of the statement

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")
3. A prover writes down a proof of the statement
4. The verifier uses an algorithm of their choice to check the statement and proof, and accepts or rejects accordingly.

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")
3. A prover writes down a proof of the statement
4. The verifier uses an algorithm of their choice to check the statement and proof, and accepts or rejects accordingly.
5. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")
3. A prover writes down a proof of the statement
4. The verifier uses an algorithm of their choice to check the statement and proof, and accepts or rejects accordingly.
5. NP as a proof system:
   - $L \subseteq \{0, 1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0, 1\}^{\text{poly}(|x|)}$

witness can be thought of as proof that x in language

Proof Systems

A proof system looks like this:

1. A prover and a verifier agree on the following:
   - The prover must provide proofs in a certain format
   - The verifier can use algorithms from a certain complexity class for verification
2. A statement is given to both prover and verifier (for instance "Graph $G(V, E)$ has a Hamiltonian Cycle")
3. A prover writes down a proof of the statement
4. The verifier uses an algorithm of their choice to check the statement and proof, and accepts or rejects accordingly.
5. NP as a proof system:
   - $L \subseteq \{0, 1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0, 1\}^{\text{poly}(|x|)}$
   - Verifier picks a poly-time Turing Machine V and outputs
     $$\begin{cases} TRUE, & \text{if } V(x, w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$$

V checks that proof is correct for input

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0,1\}^{\text{poly}(|x|)}$

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0,1\}^{\text{poly}(|x|)}$
   - Verifier picks a poly-time Turing Machine V and outputs
     $$\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$$

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
    - **Completeness:** *correct* statements *have a proof* in the system
    - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
    - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine
    - Given an element x, the prover gives a proof (also known as witness) $w \in \{0,1\}^{\text{poly}(|x|)}$
    - Verifier picks a poly-time Turing Machine V and outputs $\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$
    - **Completeness:** $x \in L \Rightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)}$ such that $V(x,w) = 1$

Correct statement   ∃ proof   Correct

Proof Systems - Completeness and Soundness

How good is a proof system?

1. Two parameters (aside from efficiency):
   - **Completeness:** *correct* statements *have a proof* in the system
   - **Soundness:** *false* statements *do not have a proof* in the system
2. NP as a proof system:
   - $L \subseteq \{0,1\}^n$ is the language, verifier can use any poly-time Turing Machine
   - Given an element x, the prover gives a proof (also known as witness) $w \in \{0,1\}^{\text{poly}(|x|)}$
   - Verifier picks a poly-time Turing Machine V and outputs $\begin{cases} TRUE, & \text{if } V(x,w) = 1 \\ FALSE, & \text{otherwise} \end{cases}$
   - **Completeness:** $x \in L \Rightarrow \exists w \in \{0,1\}^{\text{poly}(|x|)}$ such that $V(x,w) = 1$
   - **Soundness:** $x \notin L \Rightarrow \forall w \in \{0,1\}^{\text{poly}(|x|)}$ we have $V(x,w) = 0$

for every possible proof

verifier will catch that is false

Interactive Proofs: Complexity Classes

The above discussion motivates us to define complexity classes in terms of proof systems!

Interactive Proofs: Complexity Classes

The above discussion motivates us to define complexity classes in terms of proof systems!

Definition (Interactive Proof Systems)

The class IP consists of all languages L that have an interactive proof system (P, V) where

Interactive Proofs: Complexity Classes

The above discussion motivates us to define complexity classes in terms of proof systems!

Definition (Interactive Proof Systems)

The class IP consists of all languages L that have an interactive proof system (P, V) where

1. the verifier V is a randomized, polynomial time algorithm
2. there is an honest prover P (who can be all powerful)

V is efficient

P honest if $\begin{cases} x \in L & \text{then P valid proof} \\ x \notin L & \text{then P false (don't try to cheat)} \end{cases}$

Interactive Proofs: Complexity Classes

The above discussion motivates us to define complexity classes in terms of proof systems!

Definition (Interactive Proof Systems)

The class IP consists of all languages L that have an interactive proof system (P, V) where

1. the verifier V is a randomized, polynomial time algorithm
2. there is an honest prover P (who can be all powerful)
3. for any $x \in \{0,1\}^*$
   - $x \in L \Rightarrow$ for an *honest* prover P,

$$\Pr[V(x, P) = 1] = 1$$

   - $x \notin L \Rightarrow$ for *any prover* P',

$$\Pr[V(x, P') = 1] \leq 1/2$$

$x \in L \Rightarrow \exists$ honest prover s.t. V accepts

P' could be malicious

ONE-SIDED ERROR

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
3. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
3. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* consists of languages L that have a randomized poly-time verifier V such that

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
3. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$

Probabilistic Proof Systems

What if we allow our verifier to run a randomized algorithm?

Definition (Probabilistic Proof System)

In a probabilistc proof system, the verifier has a randomized algorithm V for which:

1. Given language L (the language of correct statements)
2. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
3. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
2. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Quantifying Probabilistic Proof Systems

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* (PCP) consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
2. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

Quantifying Probabilistic Proof Systems

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* (PCP) consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
2. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

- $PCP[r(n), q(n)]$ consists of all languages $L \in PCP$ such that, on inputs of length n

Quantifying Probabilistic Proof Systems

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* (PCP) consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
2. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

- $PCP[r(n), q(n)]$ consists of all languages $L \in PCP$ such that, on inputs of length n
  1. Uses $O(r(n))$ random bits
  2. Examines $O(q(n))$ bits of a proof w

V

Quantifying Probabilistic Proof Systems

Definition (Probabilistic Checkable Proofs (PCPs))

The class of *Probabilistic Checkable Proofs* (PCP) consists of languages L that have a randomized poly-time verifier V such that

1. $x \in L \Rightarrow$ there exists proof w such that $\Pr[V(x, w) = 1] = 1$
2. $x \notin L \Rightarrow$ for any proof w, we have $\Pr[V(x, w) = 1] \leq 1/2$

- $PCP[r(n), q(n)]$ consists of all languages $L \in PCP$ such that, on inputs of length n
  1. Uses $O(r(n))$ random bits
  2. Examines $O(q(n))$ bits of a proof w

don't need to look at entire proof!

Theorem (PCP theorem [AS'98, ALMSS'98])

$$PCP[\log n, 1] = NP$$

O(1)

V_L $[r_L(n), q_L(n)]$

prob. proof system

deterministic proof system

PCP and Approximability of Max SAT

Theorem

1. *The PCP theorem implies that there is an* $\varepsilon > 0$ *such that there is no polynomial time* $(1+\varepsilon)$*-approximation algorithm for Max 3SAT, unless P = NP.*
2. *Moreover, if Max 3SAT is hard to approximate within a factor of* $(1+\varepsilon)$*, then the PCP theorem holds.*

- In other words, the PCP theorem and the hardness of approximation of Max 3SAT are equivalent.

$$\bigwedge (a_1 \vee a_2 \vee \bar{a}_3)$$

PCP and Approximability of Max SAT

1. Let us assume the PCP theorem holds.
   - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
   - Let V be the $(O(\log n), q)$ verifier for L, where q is a constant

PCP and Approximability of Max SAT

1. Let us assume the PCP theorem holds.
   - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
   - Let V be the $(O(\log n), q)$ verifier for L, where q is a constant
2. We now describe a reduction from L to Max 3SAT which has a gap.

PCP and Approximability of Max SAT

1. Let us assume the PCP theorem holds.
   - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
   - Let V be the $(O(\log n), q)$ verifier for L, where q is a constant
2. We now describe a reduction from L to Max 3SAT which has a gap.
3. Given an instance x of problem L, we construct 3CNF formula φ_x with m clauses such that, for some ε we have
   - $x \in L \Rightarrow \varphi_x$ is satisfiable (satisfies m clauses)
   - $x \notin L \Rightarrow$ no assignment satisfies more than $(1 - \varepsilon) \cdot m$ clauses of φ_x

PCP and Approximability of Max SAT

$V(x, R)$

1. Let us assume the PCP theorem holds.
   - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
   - Let V be the $(O(\log n), q)$ verifier for L, where q is a constant
2. We now describe a reduction from L to Max 3SAT which has a gap.
3. Given an instance x of problem L, we construct 3CNF formula φ_x with m clauses such that, for some ε we have
   - $x \in L \Rightarrow \varphi_x$ is satisfiable
   - $x \notin L \Rightarrow$ no assignment satisfies more than $(1-\varepsilon) \cdot m$ clauses of φ_x
4. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.

small # random strings

PCP and Approximability of Max SAT

1. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.

poly(n) many functions

PCP and Approximability of Max SAT

1. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.
2. Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
   - Can be done with a CNF of size 2^q
   - Converting to 3CNF we get a formula of size $q \cdot 2^q$

Practice problem

$$\bigwedge (a_1 \vee \cdots \vee a_t) = \bigwedge (a_1 \vee a_2 \vee a_3)$$

$$\bigwedge (a_{i_1} \vee a_{i_2} \vee a_{i_3})$$

PCP and Approximability of Max SAT

1. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \dots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \dots, w_{i_q^R}) = 1$.
2. Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
   - Can be done with a CNF of size 2^q
   - Converting to 3CNF we get a formula of size $q \cdot 2^q$
3. Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above

PCP and Approximability of Max SAT

1. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \rightarrow \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.
2. Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
   - Can be done with a CNF of size 2^q
   - Converting to 3CNF we get a formula of size $q \cdot 2^q$
3. Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above
4. If $x \in L$ then there is a witness w such that $V(x, w)$ accepts for every random string R. In this case, φ_x is satisfiable!

PCP and Approximability of Max SAT

1. Enumerate all random inputs R for the verifier V.
   - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is poly(n).
   - For each R, V chooses q positions $i_1^R, \ldots, i_q^R$ and a boolean function $f_R : \{0,1\}^q \to \{0,1\}$ and accepts iff $f_R(w_{i_1^R}, \ldots, w_{i_q^R}) = 1$.
2. Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
   - Can be done with a CNF of size 2^q
   - Converting to 3CNF we get a formula of size $q \cdot 2^q$
3. Let φ_x be the 3CNF we get by putting together all the 3CNFs constructed above
4. If $x \in L$ then there is a witness w such that $V(x, w)$ accepts for every random string R. In this case, φ_x is satisfiable!
5. If $x \notin L$ then the verifier says NO for half of the random strings R.
   - For each such random string, at least one of its clauses fails
   - Thus at least $\varepsilon = \frac{1}{2 \cdot q \cdot 2^q}$ of the clauses of φ_x fails.

Digested Proof of Theorem

Digested Proof of Theorem

Conclusion

- Important to study hardness of approximation for NP-hard problems

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems
- Proof systems, in particular *Probabilistic Checkable Proofs*, allows us to get such strong reductions

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems
- Proof systems, in particular *Probabilistic Checkable Proofs*, allows us to get such strong reductions
- Many more applications in computer science and industry!
    - Program Checking (for software engineering)
    - Zero-knowledge proofs in cryptocurrencies
    - many more...

Acknowledgement

- Lecture based largely on:
  - Section's 1-3 of Luca's survey [Trevisan 2004]
  - [Motwani & Raghavan 2007, Chapter 7]
- See Luca's survey `https://arxiv.org/pdf/cs/0409043`

References I

Trevisan, Luca (2004)
Inapproximability of combinatorial optimization problems.
arXiv preprint cs/0409043 (2004).

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Arora, Sanjeev, and Shmuel Safra (1998)
Probabilistic checking of proofs: A new characterization of NP.
Journal of the ACM (JACM) 45, no. 1 (1998): 70-122.

Arora, Sanjeev, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy (1998)
Proof verification and the hardness of approximation problems.
Journal of the ACM (JACM) 45, no. 3 (1998): 501-555.