Lecture 14: Semidefinite Programming Relaxation and MAX-CUT

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Overview

Motivation - NP-hard problems

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    1. Find approximate solutions in polynomial time!
    2. Sometimes we even do that for problems in P (but we want much much faster solutions)

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- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

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  - Can we get **better** approximations using SDPs instead of ILPs?
- Yes. Today we will see Max-Cut (more generally constraint satisfaction relaxations)
- Very impressive recent theoretical developments! Unique Games Conjecture, Sum-of-Squares, and more!

Example

Maximum Cut (Max-Cut):

$G(V, E)$ graph.

Cut $S \subseteq V$ and size of cut is

$$|E(S, \overline{S})| = |\{(u, v) \in E \ \mid \ u \in S, v \notin S\}|.$$

Goal: find cut of maximum size.

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\end{aligned}$$

Example - Weighted Variant

Maximum Cut (Max-Cut):

$$G(V, E, w) \text{ weighted graph. } \sum_{e \in E} w_e = 1$$

Cut $S \subseteq V$ and weight of cut is the sum of weights of edges crossing cut.
Goal: find cut of maximum weight.

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$$
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   1. If solution to SDP is *integral* and *one-dimensional*, then it is a solution to QP and we are done
   2. If solution has *higher dimension*, then we have to devise *rounding procedure* that transforms

      high dimensional solutions $\rightarrow$ integral & 1D solutions

$$\text{rounded SDP solution value} \geq c \cdot OPT(QP)$$

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Analyzing ILP for Max-Cut

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- G complete graph $\Rightarrow OPT = \frac{1}{2} + \frac{1}{2(n-1)}$

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- Max-Cut NP-hard

Proof that $OPT(ILP) \geq 1/2$

Probabilistic method:

Pick $x_v = \begin{cases} 0 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2 \end{cases}$

$\mathbb{E}[z_{uv}] = 1/2$

$\mathbb{E}[\text{value of cut}] = \sum w_{uv} \cdot \mathbb{E}[z_{uv}]$

$= \frac{1}{2} \sum w_{uv} = \frac{1}{2}$

$\therefore$ there is integral solution that is

$\geq$ average (expectation)

Rounding Max-Cut ILP

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- This relaxation is not helpful! :(

- Why Relax & Round?
- Max-Cut SDP Relaxation and Rounding
- Conclusion
- Acknowledgements

Max-Cut

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Quadratic Program:

$$\begin{aligned} &\text{maximize} \quad \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - x_u x_v) \\ &\text{subject to} \quad x_v^2 = 1 \quad \text{for } v \in V \end{aligned}$$

SDP Relaxation [Delorme, Poljak 1993]

$G(V, E, w)$ weighted graph, $|V| = n$ and $\sum_{e \in E} w_e = 1$

Semidefinite Program:

$$\begin{aligned} \text{maximize} \quad & \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - y_u^T y_v\right) \\ \text{subject to} \quad & \|y_v\|_2^2 = 1 \quad \text{for } v \in V \\ & y_v \in \mathbb{R}^d \quad \text{for } v \in V \end{aligned}$$

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$d \leq n$

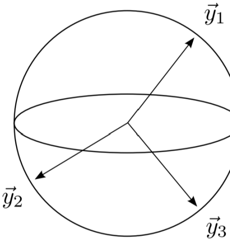
- How is that an SDP?

$X_{ij} = y_i^T y_j \quad \therefore \quad X = Y^T Y \quad Y = (y_1\ y_2 \cdots y_n)$

$X_{ii} = y_i^T y_i = \|y_i\|^2 = 1$

$\Leftrightarrow X \succeq 0$ and $X_{ii} = 1 \ \forall i \in [n]$

What is this SDP doing?



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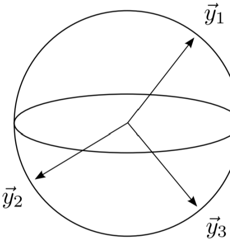

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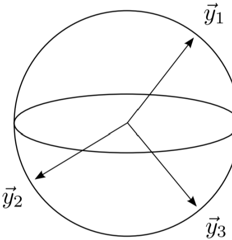

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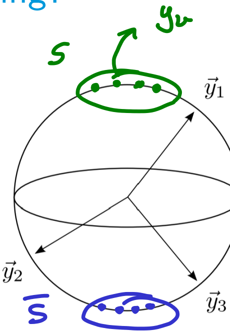

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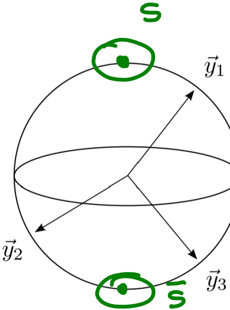

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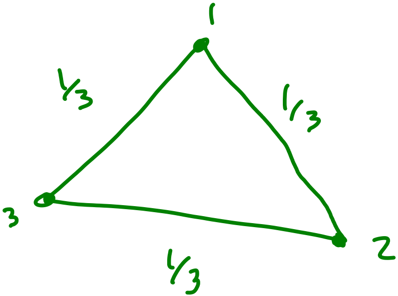
- If all y_v's are in a one-dimensional space, then we get original quadratic program

$$OPT(SDP) \geq \text{Weight of Maximum Cut}$$

Example

Let's consider $G = K_3$ with equal weights on edges.



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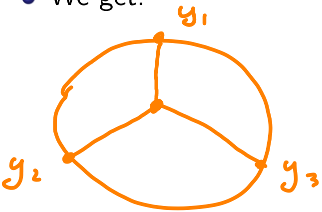

$$OPT(SDP) = \sum_{i<j} \frac{1}{3} \cdot \frac{1}{2} \cdot \left(1 - \cos(2\pi/3)\right)$$

$$= 3 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \left(1 + \frac{1}{2}\right)$$

$$= 3/4$$

$$\cos(2\pi/3) = -\cos(\pi - 2\pi/3) = -\cos(\pi/3) = \frac{1}{2}$$

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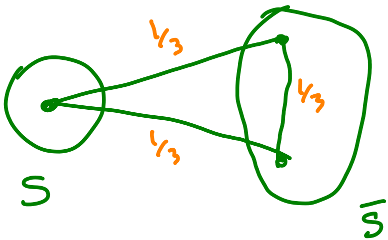

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- **Practice problem:** try this with C_5.

should get ≈ 0.88

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3. Need to “pick sides”
4. **[Goemans, Williamson 1994]:** Choose a random hyperplane though origin!
5. Choose normal vector $g \in \mathbb{R}^n$ from a Gaussian distribution.
6. Set $x_u = \text{sign}(g^T z_u)$ as our solution

Max-Cut - Rounding

1. Let $z_u \in \mathbb{R}^n$ be an optimal solution to our SDP
2. How do we convert it into a cut?
3. Need to "pick sides"
4. **[Goemans, Williamson 1994]:** Choose a random hyperplane though origin!
5. Choose normal vector $g \in \mathbb{R}^n$ from a Gaussian distribution.
6. Set $x_u = \text{sign}(g^T z_u)$ as our solution

$1 \mapsto +1$

$2, 3 \mapsto -1$

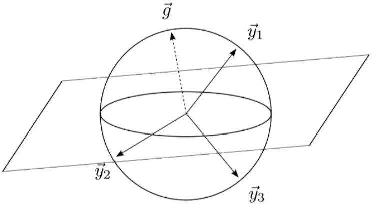

Figure 10.2: Vectors being separated by a hyperplane with normal $\vec{g}$.

Analysis of Rounding - Sketch

- Probability that edge $\{u, v\}$ crosses the cut is same as probability that z_u, z_v fall in different sides of hyperplane

$$\Pr[\{u, v\} \text{ crosses cut }] = \Pr[g \text{ splits } z_u, z_v]$$

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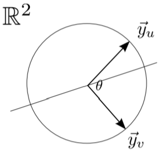

Figure 10.3: The plane of two vectors being cut by the hyperplane

Plane puts u, v different sides of cut.

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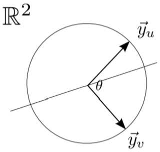

Figure 10.3: The plane of two vectors being cut by the hyperplane

- Probability of splitting z_u, z_v:

$$\Pr[\{u, v\} \text{ crosses cut}] = \frac{\theta}{\pi} = \frac{\cos^{-1}(z_u^T z_v)}{\pi} = \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

$$\mathbb{E}[\text{value of cut}] = \sum_{\{u,v\} \in E} w_{u,v} \cdot \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

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- Expected value of cut:

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$$OPT_{SDP} = \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot \left(1 - z_u^T z_v\right) = \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - \gamma_{uv})$$

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Then we have an α-approximation algorithm!

(also need to prove concentration result)

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Theorem ([Goemans, Williamson 1994])

$\alpha = 0.87856...$ *works, and gives us our approximation algorithm.*

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4. Solve SDP optimally using efficient algorithm.
   1. If solution to SDP is *integral* and *one dimensional*, then it is a solution to Max-Cut and we are done
   2. If have *higher dimensional* solutions, *rounding procedure*

      Random Hyperplane Cut algorithm, with high probability we get

$$cost(\text{rounded solution}) \geq 0.878 \cdot OPT(SDP) \geq 0.878 \cdot OPT(\text{Max-Cut})$$

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All of these are amazing final project topics!

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- Mathematical Programming hard in general
- Sometimes can get SDP rounding!
- Solve SDP and round the solution
  - Deterministic rounding when solutions are nice
  - Randomized rounding when things a bit more complicated

our rounding will "decrease dimension" and "make it integral".

Acknowledgement

- Lecture based largely on:
  - Lecture 14 of Anupam Gupta and Ryan O'Donnell's Optimization class

    `https://www.cs.cmu.edu/~anupamg/adv-approx/`
- See their notes at

  `https://www.cs.cmu.edu/~anupamg/adv-approx/lecture14.pdf`

References I

Delorme, Charles, and Svatopluk Poljak (1993)
Laplacian eigenvalues and the maximum cut problem.
Mathematical Programming 62.1-3 (1993): 557-574.

Goemans, Michel and Williamson, David 1994
0.879-approximation algorithms for Max Cut and Max 2SAT.
Proceedings of the twenty-sixth annual ACM symposium on Theory of computing. 1994