Lecture 15: Approximation Algorithms for Travelling Salesman Problem

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Overview

- Equivalent Versions of Traveling Salesman Problem
- Approximation Algorithms for Traveling Salesman Problem
- Conclusion
- Acknowledgements

Traveling Salesman Problem

- **Input:** set of points X and a symmetric distance function

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

$$d(x, y) = d(y, x) \geq 0$$

Traveling Salesman Problem

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- For any path $p_0 \to p_1 \to \cdots \to p_t$ in X, *length* of the path is sum of distances traveled

$$\sum_{i=0}^{t-1} d(p_i, p_{i+1})$$

$$d(p_0, p_1) + d(p_1, p_2) + \cdots + d(p_{t-1}, p_t)$$

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  - Efficient route planning (mail system, shuttle bus pick up and drop off...)

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- Comes in many flavours...

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$$d(x,y) \leq d(x,z) + d(z,y)$$

$\forall x,y,z \in X$.

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$$r(n) = 2^n$$

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- Solution space of Metric TSP-R is larger than solution space of Metric TSP-NR. Thus and includes solutions to Metric TSP-NR

$$\underline{OPT_R(X, d)} \leq \underline{OPT_{NR}(X, d)}$$

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Example: $a \to b \to c \to b \to d \to a \;\mapsto\; a \to b \to c \to d \to a$

C $\quad C'$

$\mathrm{cost}(C) - \mathrm{cost}(C') = d(c,b) + d(b,d) - d(c,d) \geq 0$ (METRIC)

- Let $\mathcal{C} = p_0 \to p_1 \to p_2 \to \cdots \to p_m = p_0$ be a solution to $OPT_R(X, d)$. Now, create a cycle $\mathcal{C}'$ from C simply by removing the repetitions

middle

$$a \to b \to \cdots c \to \cancel{a} \to d \to \cdots$$

becomes

$$a \to b \to \cdots \boxed{c \to d} \to \cdots$$

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(2) *Every c-approximation algorithm for Metric TSP-NR is also a c-approximation algorithm for Metric TSP-R.*

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- If we have a c-approximation algorithm for Metric TSP-NR, then we know that our solution (cycle $\mathcal{C}$) satisfies:

$$\underline{cost(C)} \leq c \cdot \underline{OPT_{NR}(X, d)}$$

$$= c \cdot OPT_R(X, d)$$

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- If we have a c-approximation algorithm for Metric TSP-NR, then we know that our solution (cycle $\mathcal{C}$) satisfies:

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- Since $OPT_{NR}(X, d) = OPT_R(X, d)$ and $\mathcal{C}$ is also a solution to Metric TSP-R, we are done.

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- Given any solution to Metric TSP-R, simply run the procedure that removes repeated visits to a vertex. This only decreases cost by metric property.

and again, we know that $\boxed{OPT_{NR}(X,d) = OPT_R(X,d)}$

Metric TSP-R equivalent to General TSP-R

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- First item follows by the fact that Metric TSP-R is a special case of General TSP-R, when the distance function satisfies the triangle inequality.

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- On input (X, d) to General TSP-R, let $G(X, E, w)$ be the complete weighted graph such that $w(x, y) = d(x, y)$. Now compute new distance $\delta : X \to \mathbb{R}_{\geq 0}$ such that

$$\delta(x, y) \leftarrow \text{ length of shortest path from } x \text{ to } y \text{ in G}$$

$E = K_X \quad (X, K_X, d) \qquad \delta(x,y) \leq d(x,y)$

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(X, δ) Metric TSP problem

$$\delta(x, y) \leftarrow \text{ length of shortest path from } x \text{ to } y \text{ in G}$$

- Note that δ satisfies triangle inequality!

$\underbrace{\text{shortest path } x \rightsquigarrow y}_{\delta(x,y)} \leq$ shortest path $x \rightsquigarrow y$ passing through z

$= \underbrace{x \rightsquigarrow z}_{\text{shortest } \delta(x,z)} + \underbrace{z \rightsquigarrow y}_{\text{shortest } \delta(z,y)}$

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- Note that δ satisfies triangle inequality!
- Give input (X, δ) to our algorithm for Metric TSP-R. Let $\mathcal{C}$ be the cycle it outputs. Thus

$$cost(\mathcal{C}) \leq c \cdot OPT_R(X, \delta)$$

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- For every pair $(x, y) \in X^2$, note that $\delta(x, y) \leq d(x, y)$, so

$$\boxed{OPT_R(X, \delta) \leq OPT_{GR}(X, d)}$$

$\Gamma = p_0 \to p_1 \to \cdots \to p_m = p_0$

$$cost_{GR}(\Gamma) = \sum_{i=0}^{m-1} d(p_i, p_{i+1})$$

$$cost_R(\Gamma) = \sum_{i=0}^{m-1} \delta(p_i, p_{i+1}) \leq cost_{GR}(\Gamma)$$

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- Let Γ be the cycle obtained from $\mathcal{C}$ by simply replacing every $x \to y$ by the shortest path $x \to p_1 \to \cdots \to p_t \to y$ in G.

$cost_R(\mathcal{C}) \leq cost_{GR}(\mathcal{C})$ wrong way

$\mathcal{C} \quad \cdots \to \boxed{x \to y} \to \cdots$

$\hookrightarrow x \to p_1 \to p_2 \to \cdots \to y$ in G

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  1. Note that

$$\boxed{\underline{cost(\mathcal{C}, \delta)} = \underline{cost(\Gamma, d)}}$$

General TSP w/ repetition

$\delta(x,y)$ = length (under d) of shortest path from x to y in X.

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$$cost(\mathcal{C}, \delta) = cost(\Gamma, d)$$

- Combining the inequalities so far, we get:

$$\underline{cost(\Gamma, d)} = \underline{cost(\mathcal{C}, \delta)} \leq \underbrace{c \cdot opt_R(X, \delta)}_{\text{metric R}} \leq c \cdot \underbrace{opt_{GR}(X, d)}_{\text{general R}}$$

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A 2-approximation algorithm

The following lemma gives us a way to get a 2-approximation algorithm:

Lemma

Let $T(X, E, d)$ be a weighted tree with vertices X and weights given by the distance function $d : X \times X \to \mathbb{R}_{geq0}$. There is a cycle $\mathcal{C}$ that reaches each vertex at least once, and such that

$$cost(\mathcal{C}, d) = \boxed{2 \cdot cost(T, d).}$$

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- Consider a DFS visit of the tree.
- Each edge traversed exactly twice

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Let $T(X, E, d)$ be a weighted tree with vertices X and weights given by the distance function $d : X \times X \to \mathbb{R}_{geq0}$. There is a cycle $\mathcal{C}$ that reaches each vertex at least once, and such that

$$cost(\mathcal{C}, d) = 2 \cdot cost(T, d).$$

- Consider a DFS visit of the tree.
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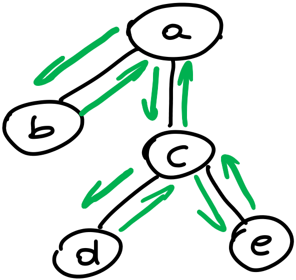
Example

Tree



DFS goes:

$a \to b \to a \to c \to d \to c \to e \to c \to a$

Edges:

$\{a,b\}$, $\{a,b\}$, $\{a,c\}$, $\{c,d\}$, $\{c,d\}$, $\{c,e\}$, $\{c,e\}$, $\{a,c\}$

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cost of minimum spanning tree $\boxed{\text{easy to get}}$

is lower bound on optimum solution

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   - Note that the weighted graph $H(X, F, d)$ is connected. Let T' be a spanning tree of this graph.

$$\underbrace{cost(T', d)}_{T' \subset F} \leq \underbrace{cost(\mathcal{C}, d)}_{\text{has all edges in } F} = OPT_{GR}(X, d)$$

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$$cost(T', d) \leq cost(\mathcal{C}, d) = OPT_{GR}(X, d)$$

   - Since T' is a spanning tree of $\underline{X}$, we have that

$$\boxed{cost(T, d)} \leq \boxed{cost(T', d)} \leq OPT_{GR}(X,d)$$

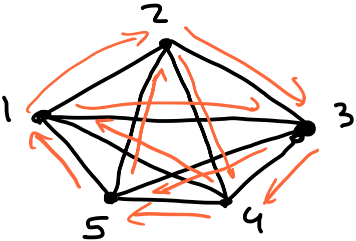
and we are done.

Eulerian Tours

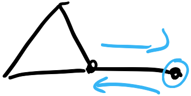
Definition (Eulerian Cycle)

An Eulerian cycle in a multigraph $G(V, E)$ is a cycle $p_0 \to p_1 \to \cdots \to p_m = p_0$ such that the number of edges $\{u, v\} \in E$ is equal to the number of times $\{u, v\}$ is used in the cycle.

In other words, each edge is used *exactly once*.



$1 \to 2 \to 3 \to 4 \to 5 \to 1 \to 3$
$\to 5 \to 2 \to 4 \to 1$



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Theorem (Eulerian Cycle Existence and Algorithm)

A multi-graph $G(V, E)$ has an Eulerian cycle if, and only if, every vertex has even degree and the vertices of positive degree are connected.

Moreover, there is a polynomial time algorithm that, on input a connected graph $G(V, E)$ in which every vertex has even degree, outputs an Eulerian cycle.

Proof of Theorem I

($\Rightarrow$) $G(V,E)$ has Eulerian cycle, $u \in V$

$\Rightarrow$ for each "incoming edge" of the cycle $\boxed{(w,u)}$, by following cycle we can pair it with the subsequent "out going edge" $\boxed{(u,w')}$. $\Rightarrow$ $\deg(u)$ even, as all edges are distinct.

Proof of Theorem II

($\Leftarrow$) Induction on # edges in graph.

If $G(V,E)$ connected and all vertices have even degree, then G has a cycle.

If every vertex has degree 2, then G must be a cycle (since it's connected). This case we are done.

Otherwise take cycle without repetition starting from vertex of degree ≥ 4. (such cycle must exist as G connected). Removing this cycle (and vertices of degree 0) we get smaller connected graph with even degrees. $\boxtimes$

Procedure above gives poly-time algorithm!

How to find small cycle? (DFS!)

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$$\sum_{v \in T} \deg(v) = \underbrace{2 \cdot |E|}_{\text{even}}$$

$$\boxed{\sum_{v \in O} \deg(v)} = 2 \cdot |E| - \underbrace{\sum \deg}_{\text{even}}$$

$$\sum_{v \in O} \underbrace{\deg(v)}_{\text{odd}} + \sum_{u \in E} \underbrace{\deg(u)}_{\text{even}}$$

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  - Thus we get a 3/2-approximation!

Putting Everything Together

use this form for simplicity

as they are equivalent!

1. **Input:** (X, d) instance of *Metric TSP-R*
2. **Output:** Cycle $\mathcal{C}$ over X covering every vertex at least once, with

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7. Find Eulerian Cycle $\mathcal{C}$ on E (it exists by thm)
8. Output $\mathcal{C}$

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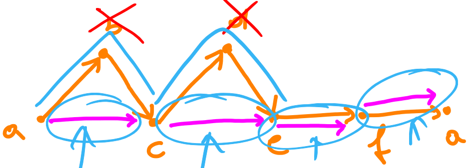

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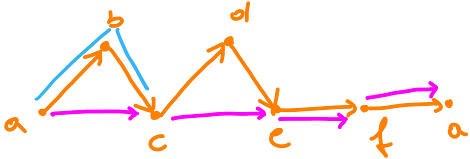

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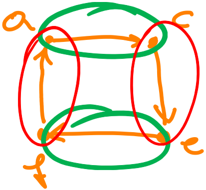

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  - Thus:

$$cost(\mathcal{M}, d) \leq \frac{1}{2} \cdot cost(C, d) \overset{\text{metric}}{\leq} \frac{1}{2} \cdot cost(\Gamma, d) \overset{\text{optimality}}{=} \frac{1}{2} \cdot OPT_R(X, d).$$

Conclusion

- Traveling Salesman Problem - important, but NP-hard
- Equivalent variants of TSP

(the first variant is NOT equivalent to other 3 because the last 3 have c-approx. algorithm and first hard to approximate)

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- Combinatorial Approximation Algorithms for TSP
- Achieve approximation algorithm by looking at an object (minimum spanning tree) which is a *lower bound* on the cost of the optimum
- This object (minimum spanning tree) is also easy to find, so exploit that to our advantage to get approximation algorithm.

Acknowledgement

- Lecture based largely on:
  - Lectures 2-4 of Luca's Optimization class
- See Luca's Lecture 3 notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture03.pdf
- See Luca's Lecture 4 notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture04.pdf