Lecture 12: Linear Programming and Duality Theorems

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Overview

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Part II
  - Game Theory
  - Learning Theory - Boosting
- Conclusion
- Acknowledgements

Mathematical Programming

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\begin{aligned}
\text{minimize} \quad & f(x) \\
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& \vdots \\
& g_m(x) \leq 0 \\
& x \in \mathbb{R}^n
\end{aligned}
$$

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- Very general family of problems.

For instance: NP-hard when g_i's are quadratic polynomials!

Question: how much harder can it get?

(much much harder!)

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- Very general family of problems.
- Special case is when all functions $f, g_1, \ldots, g_m$ are *linear* functions (called *Linear Programming* - LP for short)
- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n = c^T x$$

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$A_i \in \mathbb{R}^n$

linear inequalities

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$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \le 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$ (encoding the m inequalities)

$$\begin{pmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_m^T \end{pmatrix} (x)$$

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$$f(x) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n = c^T x$$

Linear Programming deals with problems of the form

$\max -c^T x \longleftrightarrow$

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

We can *always* represent LPs in *standard form*:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

Practice problem: show that we can always represent in standard form!

Why should I care?

- Linear Programs appear everywhere in life: many problems of interest (resource allocation problems) can be modelled as linear program!

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- Stock portfolio optimization:
  - n companies, stock of company i costs $c_i \in \mathbb{R}$
  - company i has expected profit $p_i \in \mathbb{R}$
  - our budget is $B \in \mathbb{R}$

(we allow fractional shares)

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profit from 1

$$\begin{aligned} \text{maximize} \quad & p_1 \cdot x_1 + \cdots + p_n \cdot x_n \\ \text{subject to} \quad & c_1 \cdot x_1 + \cdots + c_n \cdot x_n \leq B \\ & x \geq 0 \end{aligned}$$

total profit

↳ cost of portfolio

x_i ← amount of stock i that you want to have

Why should I care?

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- Other problems, such as *data fitting*, *linear classification* can be modelled as linear programs.

Important Questions

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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Important Questions

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   - Is there a minimum? Or is the minimum $-\infty$?
3. Can we characterize *optimality*?
   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?

Important Questions

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   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?
4. How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs?

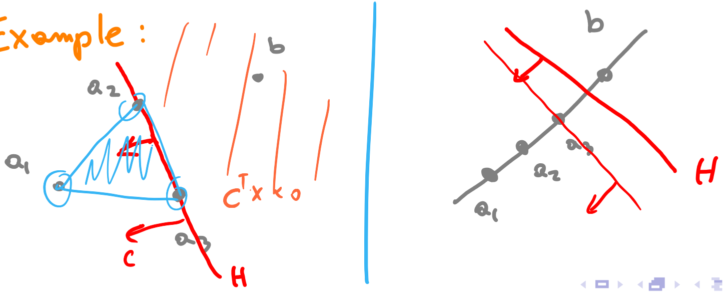
Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, and $t := \text{rank}\{a_1, \ldots, a_m, b\}$. Then either

1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or* $b = \alpha_1 a_1 + \cdots + \alpha_m a_m \quad \alpha_i \geq 0$
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
   - *H contains $t-1$ linearly independent vectors from $a_1, \ldots, a_m$*

Example :



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   - *H contains $t - 1$ linearly independent vectors from $a_1, \ldots, a_m$*

Remark

The hyperplane H above is known as the *separating hyperplane*.

Farkas' Lemma

$A = (A_1\, A_2 \cdots A_n) \quad A_i \in \mathbb{R}^m$

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$* (LP feasible)
2. *$y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$*

① ⇒ ② $\quad y^TA \geq 0 \quad Ax = b$

$y^T A x = y^T b$

LHS ≥ 0 ($y^TA \geq 0$, $x \geq 0$)

$\Rightarrow y^T b \geq 0$

② ⇒ ① (not ① ⇒ not ②)

$A_1, \dots, A_n$ columns of A

not (1) assumption

Fundamental thm of linear inequalities ⇒ $\nexists\, \vec{x} \geq \vec{0}$ s.t. $\sum_{i=1}^{n} x_i A_i = b$ (i.e. b not non-negative combination of $A_1, \dots, A_n$) then we have separating hyperplane H_y s.t. $y^T A_i \geq 0$ but $y^T b < 0$.

$H = \{z \mid y^T z = 0\}$

$\hookrightarrow Ax = b$

not ②. ☑

Farkas' Lemma

Lemma (Farkas Lemma)

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1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *$y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$*

Equivalent formulation

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$*

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Linear Programming Duality

Consider our linear program:

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\begin{aligned} y^T A \leq c^T & \Rightarrow y^T A x \leq c^T x \\ & \Rightarrow y^T b \leq c^T x \end{aligned}$$

$x \geq 0$

objective function

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$$\begin{aligned} y^T A \leq c^T & \Rightarrow y^T A x \leq c^T x \\ & \Rightarrow y^T b \leq c^T x \end{aligned}$$

- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

Linear Programming Duality

Consider the following linear programs:

Primal LP

minimize $c^T x$
subject to $Ax = b$
$x \geq 0$

Dual LP

best maximize $y^T b$ lower bound on opt
subject to $y^T A \leq c^T$

linear program!

$y^T b \leq \min c^T x$

for any y s.t. $y^T A \leq c^T$

Linear Programming Duality

Consider the following linear programs:

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$$\begin{aligned} \text{maximize} \quad & y^T b \\ \text{subject to} \quad & y^T A \leq c^T \end{aligned}$$

- From previous slide

$$y^T A \leq c^T \Rightarrow y^T b \text{ is a lower bound on value of Primal}$$

Linear Programming Duality

Consider the following linear programs:

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

- From previous slide

$$y^T A \leq c^T \Rightarrow y^T b \text{ is a lower bound on value of Primal}$$

- Thus, the optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!

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Dual LP

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & y^T A \leq c^T \end{array}$$

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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

Primal LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual LP

$$\begin{aligned} \text{maximize} \quad & y^T b \\ \text{subject to} \quad & y^T A \leq c^T \end{aligned}$$

Remarks on Duality

Primal LP

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- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!

Remarks on Duality

Primal LP | *Dual LP*

$$\alpha^* = \text{minimize} \quad c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0$$

$$\beta^* = \text{maximize} \quad y^T b \quad \text{subject to} \quad y^T A \leq c^T$$

- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!
- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.

Remarks on Duality

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

- Optimal (maximum) value of *dual LP* lower bounds the optimal (minimum) value of the *Primal LP*!
- If $\alpha^*, \beta^* \in \mathbb{R}$ are the optimal values for primal and dual, respectively.
  - We showed that when both primal and dual are feasible, we have

$$\text{max dual } = \beta^* \leq \alpha^* = \text{ min of primal}$$

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  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)

Remarks on Duality

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  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)
  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)

Remarks on Duality

Primal LP		*Dual LP*	
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  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)
- **Practice problem:** show that dual of the dual LP is the primal LP!

Remarks on Duality

Primal LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual LP

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$$\text{max dual } = \beta^* \leq \alpha^* = \text{ min of primal}$$

  - if primal *unbounded* ($\alpha^* = -\infty$) then dual *infeasible* ($\beta^* = -\infty$)
  - if dual *unbounded* ($\beta^* = \infty$) then primal *infeasible* ($\alpha^* = \infty$)
- **Practice problem:** show that dual of the dual LP is the primal LP!
- When is the above inequality tight?

Strong Duality

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
subject to	$Ax = b$	subject to	$y^T A \leq c^T$
	$x \geq 0$		

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Strong Duality

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
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	$x \geq 0$		

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\textit{max dual} = \beta^* = \alpha^* = \textit{min of primal.}$$

always equality (feasible)

Proof of Strong Duality

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If primal LP or dual LP is feasible, then

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Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\textit{max dual} = \beta^* = \alpha^* = \textit{min of primal}.$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\textit{max dual} = \beta^* = \alpha^* = \textit{min of primal}.$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$

$\varepsilon \geq 0$

$\mathbb{R}^{m+1 \times n}$

$-(\alpha^* - \varepsilon)$

$\alpha^* - \varepsilon \leq \alpha^* = \min$ in Primal

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal.}$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$
3. Apply Farkas' lemma on $Bx = v(0)$ and $x \geq 0$. This system has a solution, so we get:

$Ax = b$, $\quad -c^T x = -\alpha^*$, $\quad x \geq 0$

$(y^T \;\; z) B \leq 0 \implies (y^T \;\; z) \begin{pmatrix} b \\ -\alpha^* \end{pmatrix} \leq 0$

that is $y^T A - z c^T \leq 0 \implies y^T b - z\alpha^* \leq 0$

i.e. $\boxed{y^T A \leq z c^T} \implies \boxed{y^T b \leq z\alpha^*}$

(in particular if $z = 0$ we have $y^T A \leq 0 \Rightarrow y^T b \leq 0$.)

Proof of Strong Duality

$y \cdot \frac{1}{z} = y_\varepsilon \rightarrow$ solution to dual

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\text{max dual} = \beta^* = \alpha^* = \text{min of primal.}$$

$y_\varepsilon^T b > \alpha^* - \varepsilon \Rightarrow$ value of dual α^*

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$ $\quad Ax = b$, $c^T x = \alpha^* - \varepsilon$, $x \geq 0$
3. Apply Farkas' lemma on $Bx = v(0)$ and $x \geq 0$. This system has a solution, so we get:
4. Now, if $\varepsilon > 0$, applying Farkas' lemma on system $Bx = v(\varepsilon)$ and $x \geq 0$ we get: $Bx = v(\varepsilon)$ has no solution! ($\alpha^* - \varepsilon < \alpha^* = \min$ Primal LP)

Farkas Lemma: there is $(y^T\ z) \in \mathbb{R}^{n+1}$ s.t. $y^T A \leq c^T \cdot z$ and $y^T b > z(\alpha^* - \varepsilon)$ (by previous slide $z \neq 0$) can assume $z > 0$ (why?)

$\Rightarrow \exists\, y_\varepsilon$ s.t. $y_\varepsilon^T b > \alpha^* - \varepsilon$ $(\forall \varepsilon)$.

Proof of Strong Duality

Theorem (Strong Duality)

If primal LP or dual LP is feasible, then

$$\textit{max dual} = \beta^* = \alpha^* = \textit{min of primal}.$$

1. Let x^* be such that $c^T x^* = \alpha^*$. Can assume that $\alpha^* \neq -\infty$.
2. Let $B = \begin{pmatrix} A \\ -c^T \end{pmatrix}$ and $v(\varepsilon) = \begin{pmatrix} b \\ -\alpha^* + \varepsilon \end{pmatrix}$
3. Apply Farkas' lemma on $Bx = v(0)$ and $x \geq 0$. This system has a solution, so we get:
4. Now, if $\varepsilon > 0$, applying Farkas' lemma on system $Bx = v(\varepsilon)$ and $x \geq 0$ we get:
5. Thus, for any $\varepsilon > 0$ there is $y \in \mathbb{R}^m$ such that $y^T A \leq c^T$ and $\beta^* \geq y^T b > \alpha^* - \varepsilon$.

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \le b$$

have at least one solution, and suppose that inequality

$Ax \le b \Longrightarrow$

$$c^T x \le \delta$$

holds whenever x satisfies $Ax \le b$. Then, for some $\delta' \le \delta$ the linear inequality

$$c^T x \le \delta' \le \delta$$

is a non-negative linear combination of the inequalities of $Ax \le b$.

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Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Two-player games

Setup:

- Two players (Alice and Bob)
- Each player has a (finite) set of strategies $S_A = \{1, \ldots, m\}$ and $S_B = \{1, \ldots, n\}$

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Alice

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Women

Bob Man

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

	Football	Opera
Football	(2,1)	(0,0)
Opera	(0,0)	(1,2)

Table: Battle of the sexes payoff matrices

Nash Equilibrium

Assuming players are rational, i.e. want to maximize their payoffs, we have:

Definition (Nash Equilibrium)

A strategy profile (i, j) is called a Nash equilibrium if the strategy played by each player is optimal, *given the strategy of the other player*. That is:

1. $A_{ij} \geq A_{kj}$ for all $k \in S_A$
2. $B_{ij} \geq B_{i\ell}$ for all $\ell \in S_B$

(1) if Alice knew Bob playing j then she has no incentive to not play i

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Table: Battle of the sexes payoff matrices

	Silent	Snitch
Silent	(-1,-1)	(-10,0)
Snitch	(0,-10)	(-5,-5)

Table: Prisoner's dilemma

Mixed Strategies

Definition (Mixed Strategy)

A mixed strategy is a probability distribution over a set of pure strategies S. If Alice's strategies are $S_A = \{1, \ldots, n\}$, her mixed strategies are:

$$\Delta_A := \{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } \|x\|_1 = 1\}$$

$$\sum_{i=1}^{n} x_i = 1$$

$x_i \leftarrow$ probability of playing i

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$$v_A(x, y) = \sum_{(i,j) \in S_A \times S_B} A_{ij} x_i y_j = x^T A y$$

$$v_B(x, y) = \sum_{(i,j) \in S_A \times S_B} B_{ij} x_i y_j = x^T B y$$

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Goalie

Player

	Jump left	Jump right
kick left	(-1,1)	(1,-1)
kick right	(1,-1)	(-1,1)

Table: Penalty Kick

payoff of Goalie is higher by switching

player has higher payoff

not NE (player has incentive to deviate)

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- *Zero-Sum Game*: payoff matrices satisfy $A = -B$

$A + B = 0$

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- No pure Nash Equilibrium!
- One mixed Nash equilibrium: $x = y = (1/2, 1/2)$

Practice problem

Von Neumann's Minimax Theorem

Theorem

In a zero-sum game, for any payoff matrix $A \in \mathbb{R}^{m \times n}$*:*

$$\max_{x \in \Delta_A} \min_{y \in \Delta_B} x^T A y = \min_{y \in \Delta_B} \max_{x \in \Delta_A} x^T A y$$

Alice wants to max her payoff

Alice's payoff (Bob's payoff $B = -A$, $x^T B y = -x^T A y$)

Bob minimizes Alice's payoff

LHS : Bob picked first

RHS : Alice " "

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For given $x \in \Delta_A$: (Alice's strat.)

$$\min_{y \in \Delta_B} x^T A y = \min_{j \in S_B} (x^T A)_j$$

(pure strategy)

Bob picks j

$$(x^T A)_j \le (x^T A)_\ell$$

$$x^T A y = \sum_\ell y_\ell (x^T A)_\ell \ge \sum_\ell y_\ell \cdot (x^T A)_j = (x^T A)_j$$

($(x^T A)_\ell \ge (x^T A)_j$)

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For given $x \in \Delta_A$:

$$\min_{y \in \Delta_B} x^T A y = \min_{j \in S_B} (x^T A)_j$$

Left hand side can be written as

Alice's getting

$$\begin{aligned} \max \quad & s \\ \text{s.t.} \quad & s \leq (x^T A)_j \quad \text{for } j \in S_B \\ & \sum_{i \in S_A} x_i = 1 \\ & x \geq 0 \end{aligned}$$

← strategies of Alic

Probability distribution

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For given $y \in \Delta_B$:

$$\max_{x \in \Delta_A} x^T A y = \max_{i \in S_A} (Ay)_i$$

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For given $y \in \Delta_B$:

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These LPs form a Primal-Dual Pair!

Left hand side can be written as

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Strong duality

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- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Part II
  - Game Theory
  - Learning Theory - Boosting
- Conclusion
- Acknowledgements

Learning Theory

Consider classification problem over $\mathcal{X}$:

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distribution over elements

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- Data is sampled from unknown distribution $q \in \Delta_{\mathcal{X}}$
- *Weak learning assumption*:

For any distribution $q \in \Delta_{\mathcal{X}}$, there is a hypothesis $h \in \mathcal{H}$ which is wrong less than half the time.

$$\exists \gamma > 0, \ \forall q \in \Delta_{\mathcal{X}}, \ \exists h \in \mathcal{H}, \quad \Pr_{x \sim q}[h(x) \neq c(x)] \leq \frac{1-\gamma}{2}$$

h wrong

less than half the time

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- Surprisingly, weak learning assumption implies something much stronger: it is possible to *combine* classifiers in $\mathcal{H}$ to construct a *classifier* that is *always right* (known as *strong learning*).

Boosting

on the hypotheses

Theorem

Let $\mathcal{H}$ be a set of hypotheses satisfying weak learning assumption. Then there is distribution $p \in \Delta_{\mathcal{H}}$ such that the weighed majority classifier

$$c_p(x) := \begin{cases} 1, & \text{if } \sum_{h \in \mathcal{H}} p_h \cdot h(x) \geq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

weight

is always correct. That is, $c_p(x) = c(x)$ for all $x \in \mathcal{X}$

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- Let $M \in \{-1, 1\}^{m \times n}$, where $m = |\mathcal{X}|$ and $n = |\mathcal{H}|$.

$$M_{ij} = \begin{cases} +1, & \text{if classifier } h_j \text{ wrong on } x_i \\ -1, & \text{otherwise} \end{cases}$$

(payoff)

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- Weak learning:

$$\sum_{1 \leq i \leq n} q_i \cdot \delta_{h_j(x_i) \neq c(x_i)} \leq \frac{1 - \gamma}{2}$$

$q \in \Delta_X$

LHS: Pr[h] wrong

$\begin{cases} 1 & \text{if } h_j \text{ wrong} \\ 0 & \text{if } h_j \text{ right} \end{cases}$

Boosting - Proof

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1 if h_j wrong

-1 if h_j right

- Note that $M_{ij} = 2 \cdot \delta_{h_j(x_i) \neq c(x_i)} - 1$

$$q^T M e_j \le -\gamma \Rightarrow q^T M p \le -\gamma$$

for any $p \in \Delta_{\mathcal{H}}$.

any $p \in \Delta_{\mathcal{H}}$

$$\sum_{i=1}^{n} q_i \cdot (2\delta_{ij} - 1) \le -\gamma$$

$$\sum_{i=1}^{n} q_i = 1 \quad (q \text{ Prob. dist.})$$

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$$\sum P_j \delta_{h_j \text{ wrong}} \leq \frac{1-\gamma}{2}$$

- Note that $M_{ij} = 2 \cdot \delta_{h_j(x_i) \neq c(x_i)} - 1$

$$q^T M e_j \leq -\gamma \quad \Rightarrow \quad q^T M p \leq -\gamma$$

$$e_i^T M p \leq -\gamma$$

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  - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
  - Applications in Game Theory (minimax theorem)
  - Applications in Learning Theory (boosting)
  - many more

Acknowledgement

- Lecture based largely on:
  - Lectures 3-6 of Yarom Singer's Advanced Optimization class
  - [Schrijver 1986, Chapter 7]
- See Yarom's notes at `https://people.seas.harvard.edu/~yaron/AM221-S16/schedule.html`

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