Lecture 11: Markov Chains, Random Walks, Mixing Time, Page Rank

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Overview

- Introduction
  - Why Random Walks & Markov Chains?
  - Basics on Theory of (finite) Markov Chains
- Main Topics
  - Fundamental Theorem of Markov Chains
  - Page Rank
- Conclusion
- Acknowledgements

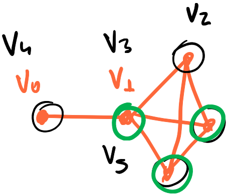
What is a Random Walk?

Given a graph $G(V, E)$

1. random walk starts from a vertex v_0
2. at each time step it moves to a *uniformly random neighbor* of the current vertex in the graph

$$v_{t+1} \leftarrow_R N_G(v_t)$$

neighbors of current vertex



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Basic questions involving random walks:

What is a Random Walk?

Given a graph $G(V, E)$ $|V| = n$ $|E| = m$

1. random walk starts from a vertex v_0 $(= 1)$
2. at each time step it moves to a *uniformly random neighbor* of the current vertex in the graph

$$v_{t+1} \leftarrow_R N_G(v_t)$$

Basic questions involving random walks:

- *Stationary distribution:* does the random walk converge to a "stable" distribution? If it does, what is this distribution?

$P_t \in \mathbb{R}^n_{\geq 0}$ probability distribution over V

$N_G(1) = \{2, 3\}$

$P_0 = (1, 0, 0, \ldots, 0)$

$P_1 = (0, 1/2, 1/2, 0, \ldots, 0)$

$\{P_t\}_{t \geq 0} \to \pi$?

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→ today's lecture

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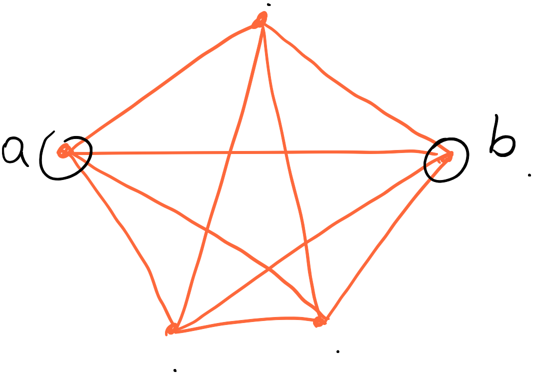
- *Hitting time:* starting from a vertex v_0, what is expected number of steps until it reaches a vertex v_f?
- *Cover time:* how long does it take to reach every vertex of the graph at least once?

Random Walk: Example

- Suppose $G(V, E) = K_n$, the complete graph, $a, b \in V$ two vertices



Random Walk: Example

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  1. What is expected number of steps to reach b in simple random walk starting at a? (i.e., hitting time)

$x_t = \Pr[\text{reach } b \text{ first time at time } t \text{ starting at } a]$

$$x_t = \left(\frac{n-2}{n-1}\right)^{t-1} \cdot \frac{1}{n-1}$$

$\left(\frac{n-2}{n-1}\right)^{t-1}$: not reaching b in first $t-1$ steps; $\frac{1}{n-1}$: reaching b at t^{th} step

$$E[\#\text{steps}] = \sum_{t \geq 1} t \cdot x_t = \sum_{t \geq 1} \frac{t}{n-1} \cdot \left(\frac{n-2}{n-1}\right)^{t-1} = \frac{1}{n-1} \sum_{t \geq 1} t \cdot \left(\frac{n-2}{n-1}\right)^{t-1}$$

(t: # steps)

$$= \frac{1}{n-1} \frac{d}{dx}\left(\sum_{t \geq 1} x^t\right)\Bigg|_{x = \frac{n-2}{n-1}} = \frac{1}{n-1} \frac{d}{dx}\left(\frac{x}{1-x}\right)\Bigg|_{x = \frac{n-2}{n-1}} = \frac{1}{n-1}\left(\frac{1}{(1-x)^2}\right)\Bigg|_{x = \frac{n-2}{n-1}}$$

$$= \frac{1}{n-1} \cdot (n-1)^2 = \boxed{n-1}$$

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Practice problem

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  2. Starting from a, what is expected number of steps to visit all vertices? (i.e, cover time)
  3. Stationary Distribution?

$$\pi = (1/n, 1/n, \ldots, 1/n)$$

$$\Pr[a \text{ in next step}] = \sum_{i=1}^{n} \Pr[\text{currently at } i] \cdot \Pr[i \to a]$$

$$= \sum_{i=1}^{n} \frac{1}{n} \cdot \left(\frac{1}{n-1} \cdot (1 - \delta_{ia}) \right) = \frac{1}{n} \cdot \frac{n-1}{n-1} = \frac{1}{n}$$

where $\frac{1}{n-1} \cdot (1-\delta_{ia})$ is $\frac{1}{n-1}$ if $i \neq a$, 0 if $i = a$

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- **Practice question:** Compare question 2 to coupon collector problem!

$(n-1) H_{n-1}$

harmonic number

What is a Markov Chain?

history of random walk

Random walk is a special kind of *stochastic process*:

$$\Pr[X_t = v_t \mid X_0 = v_0, \ldots, X_{t-1} = v_{t-1}] = \Pr[X_t = v_t \mid X_{t-1} = v_{t-1}]$$

in step t I am at vertex x_t

only useful information in the vertex at time t-1

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Process is "*forgetful*"

Markov chain is characterized by this property.

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Representing Finite Markov Chains

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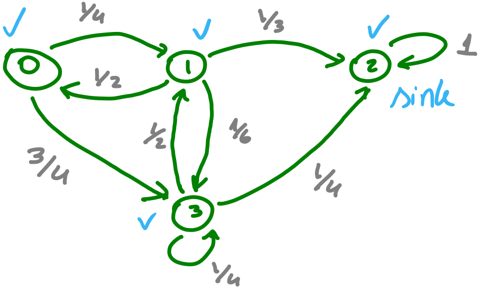
- Vertex is a state of Markov chain
- edge (i, j) corresponds to transition probability from i to j



sum weights coming out of node = 1

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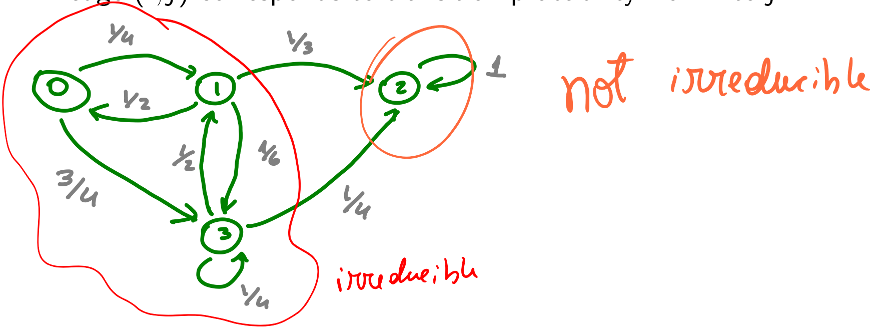
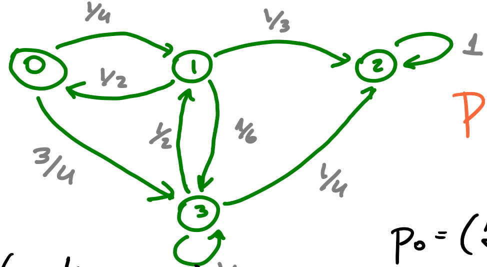


- Markov Chain *irreducible* if underlying directed graph is *strongly connected* (i.e. there is directed path from i to j for any pair $i, j \in V$)

Markov chain can be seen in weighted adjacency matrix format.



$$P = \begin{pmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}$$

$(0, 1/4, 0, 3/4)$

$p_0 = (1, 0, 0, 0)$

$p_1 = p_0 \cdot P = (0, 1/4, 0, 3/4)$

- $P \in \mathbb{R}^{n \times n}$ transition matrix — each row sums to 1 (and entries are ≥ 0)
- entry $P_{i,j}$ corresponds to transition probability from i to j
- $p_t \in \mathbb{R}^n$ probability vector: $p_t(i) :=$ Pr[being at *state i* at *time t*] — row vector
- Transition given by

$$p_{t+1} = p_t \cdot P$$

Properties of Markov Chains

- *Period* of a state i is:

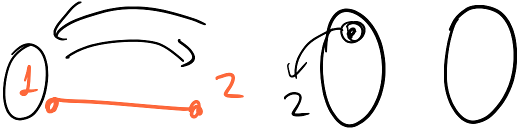
$$\gcd\{t \in \mathbb{N} \mid P_{i,i}^{t} > 0\}$$

That is, gcd of all times t such that the probability of starting at state i and being back at i at time t is positive

P_{ii}^{1} probability that I stay at i

P_{ii}^{2} probability that I $i \to v \to i$

am at i after 2 steps



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Lemma

For any finite, irreducible and aperiodic Markov Chain, there exists $T < \infty$ such that

$$P^t_{i,j} > 0 \text{ for any } i, j \in V \text{ and } t \geq T.$$

That is: at some point we will reach every vertex + positive probability of being in each vertex after walking long enough!

Stationary Distributions

Definition (Stationary Distribution)

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- Given two distributions $p, q \in \mathbb{R}^n$, their *total variational distance* is

$$\Delta_{TV}(p, q) = \frac{1}{2} \sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2} \cdot \|p - q\|_1$$

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$$\Delta_{TV}(p, q) = \frac{1}{2} \sum_{i=1}^{n} |p_i - q_i| = \frac{1}{2} \cdot \|p - q\|_1$$

- p_t *converges* to q iff $\lim_{t \to \infty} \Delta_{TV}(p_t, q) = 0$

Mixing Time of Markov Chains

Definition (Mixing Time)

The ε-mixing time of a Markov Chain is the smallest t such that

$$\Delta_{TV}(p_t, \pi) \leq \varepsilon$$

regardless of the initial starting distribution p_0.

For complete graph K_n:

transition matrix $P = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & \ddots & 0 \end{pmatrix} \cdot \frac{1}{n-1}$

eigenvalues: $\det(tI - P) = \left(\frac{1}{n-1}\right)^n \cdot \det\left([1+(n-1)t]I - J_n\right)$

all ones matrix (J_n)

eigenvalues of J_n: $\alpha_1 = n$ $\quad \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$

eigenvectors of J_n: $v_1 = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)$ $\quad v_2, \cdots, v_n$ (orthonormal)

$\therefore$ eigenvalues of P: $1 + (n-1)\lambda_i = \alpha_i$ $\quad \therefore \lambda_1 = 1$ $\quad \lambda_2 = \cdots = \lambda_n = -1/n-1$

eigenvectors of P: $v_1, v_2, \cdots, v_n$

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- Eigenvalues $\lambda_1 = 1, \lambda_2 = \cdots = \lambda_n = -1/(n-1)$, corresponding eigenvectors $v_1, \ldots, v_n$ (orthonormal)

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Fundamental Theorem of Markov Chains

- The *return time* from state i to itself is defined as

$$H_{i,i} := \min\{t \geq 1 \mid X_t = i, X_0 = i\}$$

started at i

back at i
first time
at time t

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Theorem (Fundamental Theorem of Markov Chains)

Any finite, irreducible and aperiodic Markov Chain has the following properties:

1. *There exists a unique stationary distribution π, where $\pi_i > 0$ for all $i \in [n]$*
2. *The sequence of distributions $\{p_t\}_{t \geq 0}$ will converge to π, no matter what the initial distribution is*
3. $$\pi_i = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{h_{i,i}}$$

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1. *There is unique stationary distribution* π*, where* $\pi_i > 0$ *for all* $i \in [n]$
2. *For every distribution* $p_0 \in \mathbb{R}^n_{\geq 0}$, $\lim_{t \to \infty} p_0 \cdot P^t = \pi$
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Intuition for proof of this theorem:

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Intuition for proof of this theorem:

- two random walks are "indistinguishable" after they "meet" at the *same vertex v* at a particular *time t*
- By finiteness, irreducibility and aperiodicity, two walks will meet with positive probability (and thus by Markov property) become *same distribution*

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If our underlying graph is undirected:

- If A_G adjacency matrix of $G(V, E)$ and $D = diag(d_1, d_2, \ldots, d_n)$, transition matrix:

→ deg(i)

$$D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix}$$

$P =$ [redacted] $D^{-1} \cdot A_G$

multiplies row i by $\frac{1}{d_i}$ (normalizes neighbor weights)

$$P_{t+1}(j) = \sum_{i=1}^{n} P_t(i) P_{ij} = \sum_{i=1}^{n} P_t(i) \cdot A_{ij} \cdot \frac{1}{d_i}$$

→ uniform distribution over neighbors of i

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new distribution

$$(\pi P)_j = \sum_{i=1}^{n} \pi_i \cdot P_{ij} = \sum_{i=1}^{n} \frac{d_i}{2m} \cdot A_{ij} \cdot \frac{1}{d_i}$$

j th entry

j th entry of old distribution

$$= \sum_{i=1}^{n} \frac{A_{ij}}{2m} = \frac{d_j}{2m} = \pi_j$$

$\sum_{i=1}^{n} A_{ij} = deg(j)$

$A_{ij} = 1$ iff $i \in N_G(j)$

0 otherwise

Fundamental Theorem of Markov Chains

If our underlying graph is undirected:

- In this case, easy to guess stationary distribution:

$$\pi_i = \frac{d_i}{2m}, \quad m = |E|$$

- If A_G adjacency matrix of $G(V, E)$ and $D = diag(d_1, d_2, \ldots, d_n)$, transition matrix:

$$P = D^{-1} \cdot A_G$$

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conjugate of P — symmetric

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  - This eigenvector is π!

$\pi P = \pi$

π eigenvector

$\pi_i > 0$

P has max eigenvalue 1

all other eigenvalues < 1 in absolute value

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  - All random walks converge to π, as we wanted to show.

same analysis that we did for k_n

Page Rank

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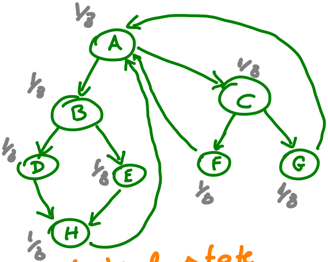
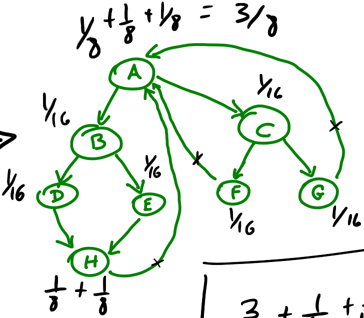
Page Rank - Example



initial state

→



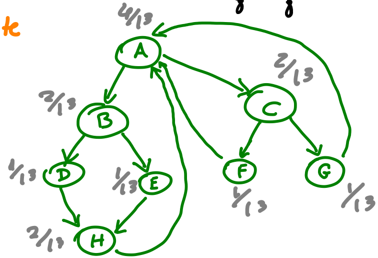
→ . . .

$$\frac{3}{8}+\frac{1}{16}+\frac{1}{16}+\frac{4}{16}+\frac{1}{4}$$

$$\frac{6+6+4}{16}=1$$

equilibrium

. . . →



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$$p_{t+1}(i) = \sum_{i:(i,j)\in E} \frac{p_t(i)}{\delta^{out}(i)} \quad \Rightarrow \quad p_{t+1} = p_t \cdot P$$

rank at time i

outgoing neighbors

$\|p_{t+1}\|_1 = \|p_t\|_1 = 1$

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- If graph finite, irreducible and aperiodic, fundamental theorem guarantees stationary distribution.

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  - Fix number $0 < s < 1$
  - Divide s fraction of its pagerank value to its neighbors,
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making it sort of complete graph

$\underline{s G} + (1-s) k_n$ satisfies FTMC

relevance

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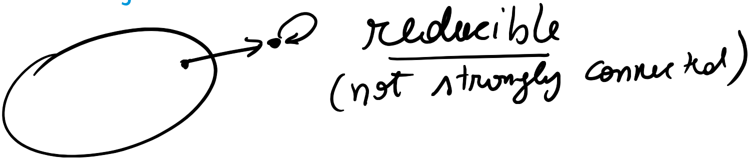
- This modification does not change "relative importance" of vertices

Conclusion

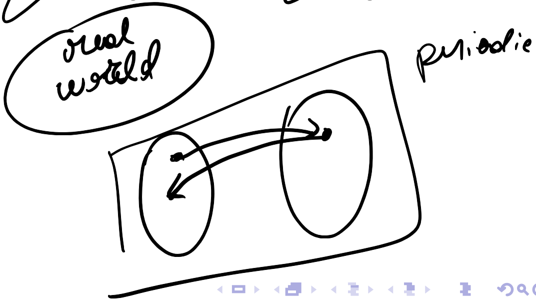
Markov Chains and Random Walks are ubiquitous in randomized algorithms.

- Page Rank algorithm (today's lecture)
- Approximation algorithms for counting problems [Karp, Luby & Madras]
  - Permanent of non-negative matrices [Jerrum, Vigoda & Sinclair]
- Sampling Problems
  - Gibbs sampling in statistical physics
  - many more places
- Probability amplification without too much randomness (efficient)
  - Random walks on expander graphs
- many more

Potential Final Projects



- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [?] and subsequently [?]
- Survey of the above, or understanding these papers is a great final project!



Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 6]
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf`
- Also see Lap Chi's notes `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf` for a proof of fundamental theorem of Markov chains for undirected graphs.

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