Lecture 10: Algebraic Techniques
Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

October 19, 2020

Overview

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)
- Cryptography

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory

Why use algebraic techniques in Computer Science?

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)
- Fingerprinting for the rescue!

[1] Think of each of them being a server of a company that deals with massive data.

Verifying String Equality

Suppose Alice and Bob each maintain the same large database of information.[1] They would like to check if their databases are *consistent*.

- Transmission of all data is expensive (*communication complexity setting*)
- Sending the entire database not feasible
- Say Alice's version of database given by bits $(a_1, \ldots, a_n)$ and Bob's version is $(b_1, \ldots, b_n)$
- Deterministic consistency check requires Alice and Bob to communicate n bits (otherwise adversary would know how to change database to make check fail)
- Fingerprinting for the rescue!

Communication complexity setting, randomized algorithms, need to work with high probability.

[1] Think of each of them being a server of a company that deals with massive data.

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
   2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
    1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
    2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
   2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)
- if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ then protocol always right

Verifying string equality

Want to check whether strings $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ equal.

Fingerprinting mechanism:

1. Let $a = \sum_{i=1}^{n} a_i \cdot 2^{i-1}$ and $b = \sum_{i=1}^{n} b_i \cdot 2^{i-1}$
2. Let $F_p(x) = x \mod p$ be a fingerprinting function, for a prime p
3. Protocol:
   1. Alice picks a random prime p and sends $(p, F_p(a))$ to Bob
   2. Bob checks whether $F_p(a) \equiv F_p(b) \mod p$, sends

$$\begin{cases} 1, & \text{if the values are equal} \\ 0, & \text{otherwise} \end{cases}$$

- **Total bits communicated:** $O(\log p)$ bits (dominated by Alice's message)
- if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ then protocol always right
- what happens when they are different?

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
    1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
    2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers

Verifying string equality

- If $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \ldots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers
- Choosing p among the first $tn\log(tn)$ primes we have that

$$\Pr[F_p(a) \not\equiv F_p(b)] \leq \frac{n}{tn\log tn/\log(tn\log tn)} = \tilde{O}(1/t)$$

Verifying string equality

- If $(a_1, \dots, a_n) = (b_1, \dots, b_n)$, then $a \neq b$.
- For how many primes can $F_p(a) \equiv F_p(b)$? (i.e., protocol will fail)
- If a number M is in $\{-2^n, \dots, 2^n\}$, then number of distinct primes $p \mid M$ is $< n$.
  1. Each prime divisor of M is ≥ 2, so if M has t distinct prime divisors, then $|M| > 2^t$
  2. $|M| \leq 2^n \Rightarrow t \leq n$
- $F_p(a) \equiv F_p(b)$ if, and only if, $p \mid a - b$.
- Thus, protocol fails for at most n choices of p
- **Prime number theorem**: there are $m/\log m$ primes among first m positive integers
- Choosing p among the first $tn \log(tn)$ primes we have that

$$\Pr[F_p(a) \not\equiv F_p(b)] \leq \frac{n}{tn \log tn / \log(tn \log tn)} = \tilde{O}(1/t)$$

- Number of bits sent is $O(\log t + \log n)$. Choosing $t = n$ solves it.

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal

In string equality, we had

$$P_A(x) = \sum_{i=1}^{n} a_i x^{i-1} \qquad P_B = \sum_{i=1}^{n} b_i x^{i-1}$$

where $a_i, b_i \in \{0, 1\}$ ($\therefore P_A(2) \neq P_B(2)$ iff $\bar{a} \neq \bar{b}$)

wanted $p \in \mathbb{N}$ prime s.t. $P_A(2) \neq P_B(2) \bmod p$

with more complicated polynomials we may not know whether $P_A(t) \neq P_B(t)$ for some value of t.

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
    1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ by FFT

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ by FFT
- Polynomial evaluation: $O(n)$

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ by FFT
- Polynomial evaluation: $O(n)$
- Can we check whether $P_1(x) \cdot P_2(x) = P_3(x)$ in $O(n)$ time?

Polynomial Identity Testing

Technique for string equality testing can be generalized to following setting:

- **Input:** "Given" two polynomials $P(x), Q(x)$, are they equal?
- Two polynomials are equal $\Leftrightarrow$ all their coefficients are equal
- So why not just compare their coefficients?
  1. Sometimes polynomials are given *implicitly* (i.e., not by their list of coefficients)
  2. $P_1(x), P_2(x), P_3(x)$, test whether: $P_1(x) \cdot P_2(x) = P_3(x)$?
  3. If P_1, P_2 have degree $\leq n$, then $\deg(P_3) \leq 2n$ (otherwise problem is trivial)
- Multiplication of two polynomials of degree n: $O(n \log n)$ by FFT
- Polynomial evaluation: $O(n)$
- Can we check whether $P_1(x) \cdot P_2(x) = P_3(x)$ in $O(n)$ time?

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

"Proof:" $\mathbb{F}[x]$ is Euclidean domain (so is $\overline{\mathbb{F}}[x]$)
(i.e. "there is division with remainder algorithm")
then induction on degree.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

Polynomial Identity Testing

Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d roots in $\overline{\mathbb{F}}$.

- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It had degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

- Can amplify probability by running multiple times or by choosing larger set S.

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ be a field and $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be a nonzero polynomial of degree $\leq d$. Then for any set $S \subseteq \overline{\mathbb{F}}$, we have:

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid a_i \in S] \leq \frac{d}{|S|}$$

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ be a field and $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ be a nonzero polynomial of degree $\leq d$. Then for any set $S \subseteq \overline{\mathbb{F}}$, we have:

$$\Pr[P(a_1, \ldots, a_n) = 0 \mid a_i \in S] \leq \frac{d}{|S|}$$

Proof by induction in number of variables.

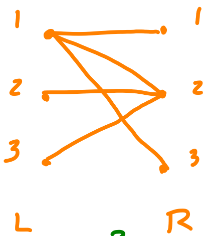
Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?

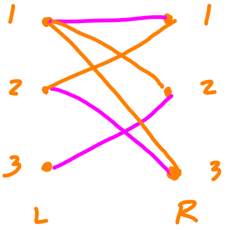
[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?



$$
\begin{array}{cc|ccc}
 & & & R & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 1 & 1 & 1 \\
L & 2 & 0 & 1 & 0 \\
 & 3 & 0 & 1 & 0
\end{array}
$$



$$
\begin{array}{cc|ccc}
 & & & R & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 1 & 1 & 1 \\
L & 2 & 1 & 0 & 1 \\
 & 3 & 0 & 1 & 0
\end{array}
$$

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

- $$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

-
$$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

- G has perfect matching $\Leftrightarrow \det(X)$ is a *non-zero polynomial*![2]

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

-

$$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$

- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]
- Testing if G has a perfect matching is a *special case* of *Polynomial Identity Testing*!

[2] First proved by Edmonds.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?
- Let $X \in \mathbb{F}^{n \times n}$ be such that

$$X_{i,j} = \begin{cases} y_{i,j}, & \text{if there is edge between } (i,j) \in L \times R \\ 0, & \text{otherwise} \end{cases}$$

- $$\det(X) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} X_{i\sigma(i)}$$
- G has perfect matching $\Leftrightarrow$ $\det(X)$ is a *non-zero polynomial*![2]
- Testing if G has a perfect matching is a *special case* of *Polynomial Identity Testing*!
- **Algorithm:** evaluate $\det(X)$ at a random value for the variables $y_{i,j}$.

[2]First proved by Edmonds.

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?

General Matching

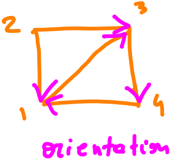
- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?
- **Tutte Matrix:** T_G is the following $2n \times 2n$ matrix: let F be an arbitrary orientation of edges in E. Then,

$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

$(x_{ij} = x_{ji})$



$$T_G = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & -x_{12} & x_{13} & -x_{14} \\ 2 & x_{12} & 0 & x_{23} & 0 \\ 3 & -x_{13} & -x_{23} & 0 & x_{34} \\ 4 & x_{14} & 0 & -x_{34} & 0 \end{array}$$

General Matching

- Ok, bipartite matching is easy (we know many algorithms for it...) what about the general case?
- **Input:** (undirected) graph $G(V, E)$ where $|V| = 2n$.
- **Output:** does G have a perfect matching?
- **Tutte Matrix:** T_G is the following $2n \times 2n$ matrix: let F be an arbitrary orientation of edges in E. Then,

$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

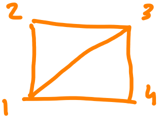
- $$\det(T_G) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- $$\det(T_G) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$$
- Each permutation $\sigma \in S_n$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{n}$.



$$\sigma = (1\ 2\ 3\ 4) \longrightarrow F_\sigma = \{(1,2),(2,3),(3,4),(4,1)\}$$

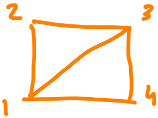
$$\underbrace{\pi = (14)(23)}_{\text{cycle decomposition of permutation}} \longrightarrow F_\pi = \{(1,4),(4,1),(2,3),(3,2)\}$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- $$\det(T_G) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$$
- Each permutation $\sigma \in S_n$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{n}$.
- Each vertex in H_σ has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.



$\sigma = (1\ 2\ 3\ 4) \longrightarrow F_\sigma = \{(1,2), (2,3), (3,4), (4,1)\}$

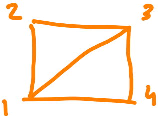
$\tilde{\pi} = (14)(23) \longrightarrow F_{\tilde{\pi}} = \{(1,4), (4,1), (2,3), (3,2)\}$

cycle decomposition of permutation

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.



$\sigma = (1\ 2\ 3\ 4) \rightarrow F_\sigma = \{(1,2),(2,3),(3,4),(4,1)\}$

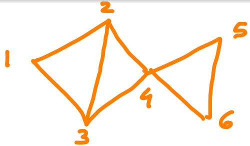
$\pi = \underbrace{(14)(23)}_{\text{cycle decomposition of permutation}} \rightarrow F_\pi = \{(1,4),(4,1),(2,3),(3,2)\}$

- Each permutation $\sigma \in S_n$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i)\}_{i=1}^n$.
- If σ only has even cycles, then H_σ gives us a perfect matching (by taking every other edge of the graph H_σ, ignoring orientation)

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.



$\sigma = (123)(456)$

$r(\sigma) = (321)(456)$

- Each permutation $\sigma \in S_n$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i)\}_{i=1}^{n}$.

$r(r(\sigma)) =$

- Otherwise, for each $\sigma \in S_n$ (that has odd cycle), there is another permutation $r(\sigma) \in S_n$ that is obtained by reversing odd cycle of H_σ containing vertex with *minimum index*.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

$$G \text{ has a perfect matching} \Leftrightarrow \det(T_G) \neq 0.$$

- Comparing $(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{n} x_{i,\sigma(i)} = -\prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$$

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^\sigma \prod_{i=1}^n [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^n [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^\sigma = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^n [T_G]_{i,\sigma(i)} = \prod_{i=1}^n x_{i,\sigma(i)} = -\prod_{i=1}^n [T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{n} x_{i,\sigma(i)} = -\prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!
- Since $r(r(\sigma)) = \sigma$, all such terms cancel!

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Comparing $(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)}$ and $(-1)^{r(\sigma)} \prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$
  - $(-1)^{\sigma} = (-1)^{r(\sigma)} \Leftarrow$ cycles of same size
  - $$\prod_{i=1}^{n} [T_G]_{i,\sigma(i)} = \prod_{i=1}^{n} x_{i,\sigma(i)} = -\prod_{i=1}^{n} [T_G]_{i,r(\sigma)(i)}$$
  - These two terms *cancel*!
- Since $r(r(\sigma)) = \sigma$, all such terms cancel!
- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)

Proof of Tutte's Theorem

Theorem (Tutte 1947)

G has a perfect matching $\Leftrightarrow \det(T_G) \neq 0$.

- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)
- If T_G has a matching, say, $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$, then take permutation $\sigma = (1\ 2)(3\ 4)\cdots(2n-1\ 2n)$

$$(-1)^{\sigma} \prod_{i=1}^{n} [T_G]_{i,\sigma(i)} = (-1)^{n} \prod_{i=1}^{n} -x_{i\sigma(i)}^2 = \prod_{i=1}^{n} x_{i\sigma(i)}^2.$$

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point
- Ore-Schwartz-Zippel-deMillo-Lipton lemma tells us that this algorithm succeeds *with high probability*

Where are my parallel algorithms?

We have seen randomized algorithms for bipartite and non-bipartite matching.
Why did you say parallel algorithms?

- The algorithms for matching consisted of:
  - testing whether a certain determinant is non-zero
  - by evaluating it at a random point
- Ore-Schwartz-Zippel-deMillo-Lipton lemma tells us that this algorithm succeeds *with high probability*
- In lecture 21, we will see that we can

compute the determinant efficiently in parallel

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!
- Can use it to compute minimum weight perfect matching (see Lap Chi's notes)

Isolation Lemma

Often times in parallel computation, when solving a problem with *many possible solutions*, it is important to make sure that *different processors* are working towards *same solution*.

Need to *single out* (i.e. isolate) a specific solution *without knowing* any element of the solution space. How to do this?

- **Solution:** Implicitly choose a *random order* on the feasible solutions and require processors to find solution of *lowest rank* in this order
- Applications also in distributed computing (breaking deadlocks)!
- Can use it to compute minimum weight perfect matching (see Lap Chi's notes)

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$*.*

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$*.*

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$, *if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$, if we assign a random weight function $w : [n] \to [2n]$ then the probability that there is a unique minimum weight set is at least $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

Isolation lemma

Lemma (Isolation Lemma)

Given a set system over $[n] := \{1, 2, \ldots, n\}$*, if we assign a random weight function* $w : [n] \to [2n]$ *then the probability that there is a unique minimum weight set is at least* $1/2$.

Example for $n = 4$:

- Set system: $S_1 = \{1, 4\}, S_2 = \{2, 3\}, S_3 = \{1, 2, 3\}$
- Random weight function $w : [4] \to [8]$ given by $w(1) = 3, w(2) = 5, w(3) = 8, w(4) = 4$
- Random weight function $w' : [4] \to [8]$ given by $w'(1) = 5, w'(2) = 1, w'(3) = 7, w'(4) = 3$

Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is *only one* set of minimum weight.

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let

$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain* v, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain* v
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain* v, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain* v
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous

Proof of Isolation lemma

1. Let $\mathcal{S}$ be our set system and $v \in [n]$.
2. Let $\mathcal{S}_v$ family of sets from $\mathcal{S}$ which *contain v*, and $\mathcal{N}_v$ the family of sets from $\mathcal{S}$ which *do not contain v*
3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.

Proof of Isolation lemma

3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$

Proof of Isolation lemma

3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$
9. If two different sets A, B have minimum weight, then any element in $A \Delta B$ must be ambiguous.

Proof of Isolation lemma

3. Let
$$\alpha_v := \min_{A \in \mathcal{N}_v} w(A) - \min_{B \in \mathcal{S}_v} w(B \setminus \{v\})$$
4. $\alpha_v < w(v) \Rightarrow v$ does not belong to any minimum weight set
5. $\alpha_v > w(v) \Rightarrow v$ belongs to every minimum weight set
6. $\alpha_v = w(v) \Rightarrow v$ is ambiguous
7. α_v is *independent* of $w(v)$, and $w(v)$ chosen uniformly at random from $[2n]$.
8. $\Pr[v \text{ ambiguous}] \leq 1/2n \Rightarrow_{\text{union bound}} \Pr[\exists \text{ ambiguous element}] \leq 1/2$
9. If two different sets A, B have minimum weight, then any element in $A \Delta B$ must be ambiguous.
10. Probability that this happens is $\leq 1/2$. (step 8)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
  - Applications in hardness of approximation!
  - Applications in blockchain (Zcash for instance)
  - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 7]
  - [Korte & Vygen 2012, Chapter 10].
- See Lap Chi's notes at
  `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf`

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Korte, Bernhard and Vygen, Jens (2012)
Combinatorial optimization. Vol. 2. Heidelberg: Springer.

Fenner, Stephen and Gurjar, Rohit and Thierauf, Thomas (2019)
Bipartite perfect matching is in quasi-NC.
SIAM Journal on Computing

Svensson, Ola and Jakub Tarnawski (2017)
The matching problem in general graphs is in quasi-NC.
IEEE 58th Annual Symposium on Foundations of Computer Science