Directed Isoperimetry and Monotonicity Testing: A Dynamical Approach

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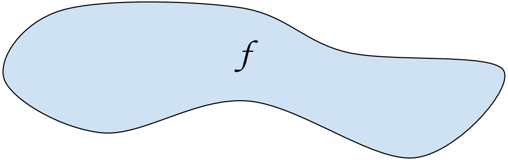
$$\|f - f_\Omega\|_{L^p(\Omega)} \le C\|\nabla f\|_{L^p(\Omega)} .$$

$\left(\int_\Omega |f(x) - f_\Omega|^p \, \mathrm{d}x\right)^{1/p}$ "distance to constant"

$\left(\int_\Omega |\nabla f(x)|^p \, \mathrm{d}x\right)^{1/p}$ "local violations"



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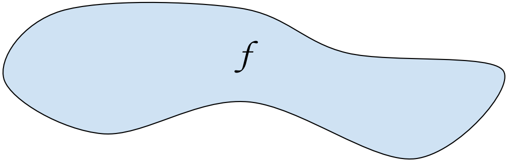

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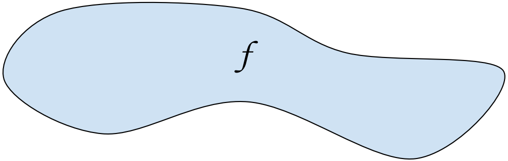

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- An isoperimetric statement: when $p = 1$, C characterizes the isoperimetric constant of Ω.
- Wide-ranging connections: mathematical physics, geometry, probability theory, diffusion processes, optimal transport...

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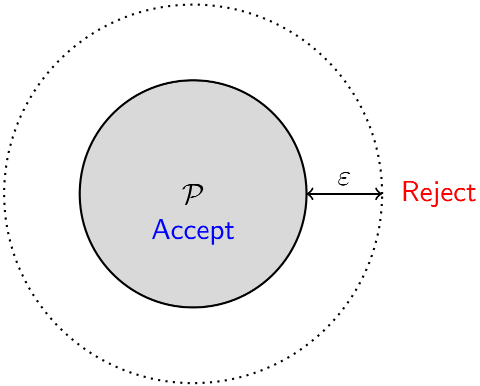
Property testing

For universe $\mathcal{X}$ and *property* $\mathcal{P} \subset \mathcal{X}$, design algorithm which, with high probability,

1. **Accepts** objects in $x \in \mathcal{P}$;
2. **Rejects** ε-far objects, i.e. when $\mathrm{dist}(x, \mathcal{P}) \geq \varepsilon$.

Goal: make decision using few queries to x (**query complexity**).



Monotonicity testing

The monotonicity property in partially ordered domain: $f(x) \leq f(y)$ whenever $x \preceq y$.

A central topic in property testing. Studied in many settings:

- Boolean case: $f : \{0,1\}^d \to \{0,1\}$ [Goldreich-Goldwasser-Lehman-Ron'98,Raskhodnikova'99, Dodis-Goldreich-Lehman-Raskhodnikova-Ron-Samorodnitsky'99, Goldreich-Goldwasser-Lehman-Ron-Samorodnitsky'00, Chakrabarty-Seshadhri'16, Chen-Servedio-Tan'14, Khot-Minzer-Safra'18, ...]
- More generally, $f : [n]^d \to \mathbb{R}$ [Goldreich-Goldwasser-Lehman-Ron'98,Raskhodnikova'99, Dodis-Goldreich-Lehman-Raskhodnikova-Ron-Samorodnitsky'99, Goldreich-Goldwasser-Lehman-Ron-Samorodnitsky'00, Chakrabarty-Seshadhri'14, Black-Chakrabarty-Seshadhri'23, Black-Kalemaj-Raskhodnikova'24, ...]
- General posets [Fischer-Lehman-Newman-Raskhodnikova-Rubinfeld-Samorodnitsky'02, Lange-Rubinfeld-Vasilyan'22]
- Continuous setting: $f : [0,1]^d \to \mathbb{R}$ [F'23, **this work**]
- Also many works on lower bounds [Blais-Raskhodnikova-Yaroslavtsev'14, Chakrabarty-Seshadhri'14, Chen-Servedio-Tan'14, Belovs-Blais'16, Chen-Waingarten-Xie'17, ...]

We focus our discussion on the **Boolean** and **continuous** settings.

Warm-up: Boolean case and the edge tester

Consider the case $f : \{0,1\}^d \to \{0,1\}$. Most natural tester:

1. Sample uniformly random edge (x, y).
2. Reject if $f(x) > f(y)$.

This is the **edge tester**. How many queries to reject ε-far f w.p. $2/3$?

Lemma (Raskhodnikova'99, Dodis-Goldreich-Lehman-Raskhodnikova-Ron-Samorodnitsky'99, Goldreich-Goldwasser-Lehman-Ron-Samorodnitsky'00)

Suppose $f : \{0,1\}^d \to \{0,1\}$ is ε-far from monotone. Let $V(f)$ denote the number of violating edges in f. Then

$$V(f) \geq \varepsilon \cdot 2^d .$$

So the tester succeeds with probability $\Omega\left(\frac{\varepsilon \cdot 2^d}{d \cdot 2^d}\right) = \Omega(\varepsilon/d) \implies O(d/\varepsilon)$ queries. ☺

Claim: a Poincaré inequality is hiding here!

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- Put another way: $\mathrm{dist}_p^{\mathrm{const}}(f)^p \lesssim \mathbb{E}\,[|\nabla f|^p]$.
- For monotonicity in the Boolean case: let ∇f be the discrete gradient of f, and let $\nabla^- f := \min\{\nabla f, 0\}$ be its directed (discrete) gradient.
- Then the lemma says: $\mathrm{dist}_1^{\mathrm{mono}}(f) \lesssim \mathbb{E}\,[\|\nabla^- f\|_1]$.

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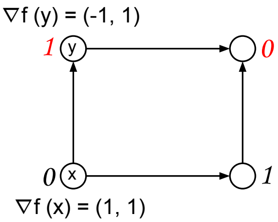

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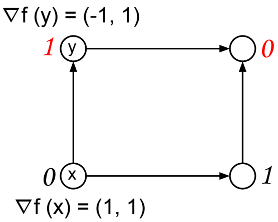

Directed isoperimetry

For $p, q \geq 1$, a classical (L^p, ℓ^q)-Poincaré inequality says

$$\mathrm{dist}_p^{\mathrm{const}}(f)^p \lesssim \mathbb{E}\left[\|\nabla f\|_q^p\right] ,$$

and a directed (L^p, ℓ^q)-Poincaré inequality says

$$\mathrm{dist}_p^{\mathrm{mono}}(f)^p \lesssim \mathbb{E}\left[\|\nabla^- f\|_q^p\right] ,$$

where $\nabla^- := \min\{0, \nabla\}$ and

$$\mathrm{dist}_p^{\mathrm{const}}(f) := \inf\left\{\|f - g\|_{L^p} : g \in L^p \text{ constant}\right\} ,$$
$$\mathrm{dist}_p^{\mathrm{mono}}(f) := \inf\left\{\|f - g\|_{L^p} : g \in L^p \text{ monotone}\right\}$$

A connection between **local violations** and the **distance** to the property of interest.

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Directed $\mathrm{dist}_p^{\mathrm{mono}}(f)^p \lesssim \mathbb{E}\left[\|\nabla^- f\|_q^p\right]$	• [Khot-Minzer-Safra'18] $(L^1, \ell^2) \implies \widetilde{O}(\sqrt{d}/\varepsilon^2)$ • [Chakrabarty-Seshadhri'16] Margulis $\implies o(d)$ • [edge tester] $(L^1, \ell^1) \implies O(d/\varepsilon)$	• [this work] $(L^2, \ell^2) \implies \widetilde{O}(\sqrt{d}M^2/\varepsilon^2)$ • [F'23] $(L^1, \ell^1) \implies O(dM/\varepsilon)$
Classical $\mathrm{dist}_p^{\mathrm{const}}(f)^p \lesssim \mathbb{E}\left[\|\nabla f\|_q^p\right]$	• [Talagrand'93] (L^1, ℓ^2) • [Margulis'74] • classical $(L^2, \ell^2) \implies (L^1, \ell^1)$	• classical (L^2, ℓ^2) • [Bobkov-Houdré'97] $(L^1, \ell^2) \implies (L^1, \ell^1)$

Conceptually similar results for: hypergrid [Black-Chakrabarty-Seshadhri'22,Braverman-Khot-Kindler-Minzer'22,Black-Chakrabarty-Seshadhri'23], real-valued functions on the Boolean cube [Black-Kalemaj-Raskhodnikova'24].

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This work

1. **Guiding question** [F'23]: is there a $o(d)$ monotonicity tester in the continuous setting?

2. **Conceptual motivation**: answering this question should require a deeper understanding of the connection between classical and directed isoperimetry.

Our answers:

1. **Yes**: in fact, roughly $\sqrt{d}$ queries suffice.

2. Main insight: a common phenomenon underlying classical and directed Poincaré inequalities in continuous space, namely **convergence properties of a partial differential equation (PDE)** – the (classical or directed) heat equation.

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Goals of this talk

1. Give the idea behind the $\widetilde{O}(\sqrt{d}M^2/\varepsilon^2)$ tester, using the directed Poincaré inequality.
2. Overview the proof ideas underlying the directed Poincaré inequality.
3. Provide some intuition for the directed heat equation, the main new ingredient toward proving the directed Poincaré inequality.

Testing model

In the continuous setting, need to define the model carefully. Adopt the L^p testing model of [Berman-Raskhodnikova-Yaroslavtsev'14]. Regularity assumption as in [F'23]: **Lipschitz**.

Input: M-Lipschitz $f : [0,1]^d \to \mathbb{R}$. Accept if f is monotone, reject if $\mathrm{dist}_2^{\mathrm{mono}}(f) \geq \varepsilon$.

Oracle queries:

- Value query given x, obtain $f(x)$.
- Directional derivative query given x, v, obtain $\nabla_v f(x) = v \cdot \nabla f(x)$.

[F'23]: tester with query complexity $O(dM/\varepsilon)$.[1] Left open: **sublinear** tester?

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Results

Theorem (Directed Poincaré inequality)

There exists a universal constant $C > 0$ such that, for all $f \in H^1((0,1)^d)$,

$$\mathrm{dist}_2^{\mathrm{mono}}(f)^2 \le C\,\mathbb{E}\left[\|\nabla^- f\|_2^2\right] .$$

Theorem (Monotonicity tester)

There exists a nonadaptive, directional derivative L^2 monotonicity tester for M-Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ with query complexity $\widetilde{O}(\sqrt{d}M^2/\varepsilon^2)$ and one-sided error.

Also, the $\sqrt{d}$ dependence is optimal among a natural generalization of *pair testers* to the continuous setting.

Idea of the tester

Tester: sample uniform $\boldsymbol{x} \in [0,1]^d$, $\boldsymbol{v} \sim \mathcal{D}$, and reject if $\boldsymbol{v} \cdot \nabla f(\boldsymbol{x}) < 0$.

Lemma (Detecting negative entries with subset sums)

There exists a distribution $\mathcal{D}$ over $\{0,1\}^d$ and universal constant $c > 0$ such that, for any nonzero $u \in \mathbb{R}^d$, we have

$$\mathop{\mathbb{P}}_{\boldsymbol{v}\sim\mathcal{D}}[u \cdot \boldsymbol{v} < 0] \geq c \cdot \frac{\|u^-\|_2^2}{\sqrt{d}\log(d) \cdot \|u\|_2^2}.$$

Essentially a *signed* version of the **group testing** problem [Dorfman'43].

Rejection probability when $\mathrm{dist}_2^{\mathrm{mono}}(f) \geq \varepsilon$:

$$\mathop{\mathbb{E}}_{\boldsymbol{x}}\left[\mathop{\mathbb{P}}_{\boldsymbol{v}}[\boldsymbol{v} \cdot \nabla f(\boldsymbol{x}) < 0]\right] \gtrsim \mathop{\mathbb{E}}_{\boldsymbol{x}}\left[\frac{\|\nabla^- f(\boldsymbol{x})\|_2^2}{\sqrt{d}\log(d)\|\nabla f(\boldsymbol{x})\|_2^2}\right] \geq \frac{1}{\sqrt{d}\log(d)M^2}\mathbb{E}\left[\|\nabla^- f\|_2^2\right]$$

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Intuition for the lemma

Lemma (Detecting negative entries with subset sums)

There exists a distribution $\mathcal{D}$ *over* $\{0,1\}^d$ *and universal constant* $c > 0$ *such that, for any nonzero* $u \in \mathbb{R}^d$*, we have*

$$\mathop{\mathbb{P}}_{\boldsymbol{v} \sim \mathcal{D}}[u \cdot \boldsymbol{v} < 0] \geq c \cdot \frac{\|u^-\|_2^2}{\sqrt{d}\log(d) \cdot \|u\|_2^2}.$$

- For simplicity, suppose $\|u\|_2^2 \approx \|u^-\|_2^2 \approx 1$.
- If $|\mathrm{supp}(u^-)| \gtrsim \sqrt{d}$, guessing singleton $v_i = 1$ works.
- Suppose $|\mathrm{supp}(u^-)| = 1$, so $u_i \approx -1$ for some i, and $u_j \approx 1/\sqrt{d}$ for $j \neq i$.
  - Guessing $v \in \{0,1\}^d$ with $\|v\|_1 \approx \sqrt{d}$ works.
- General case: guess expected $\|v\|_1$ from $1, 2, 4, \ldots, d$, and argue via *thresholds* of the form $u_i < -\tau$ that one of these granularities works.

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Analogy with the path tester

Our tester is the natural continuous analogue of the $\widetilde{O}(\sqrt{d})$ *path tester* of [KMS'18].

- [KMS'18]: sample $x \preceq y$ in $\{0,1\}^d$ joined by a path of length 2^k, for k sampled uniformly from $[\log d]$.
- For us, taking edge i in the path tester roughly corresponds to sampling $v_i = 1$ for the directional derivative query.

Directed Poincaré inequality – proof overview

Directed Poincaré inequality:

$$\mathrm{dist}_2^{\mathrm{mono}}(f)^2 \lesssim \mathbb{E}\left[\|\nabla^- f\|_2^2\right] .$$

Main ideas:

1. Introduce and study a 1D dynamical process, the directed heat equation.
2. Use it to to prove the 1D *transport-energy inequality*

$$W_2^2(\mu, \mu_\infty) \lesssim \int_{[0,1]} (\partial_x^- u)^2 \, \mathrm{d}x .$$

3. *Tensorize* the transport-energy inequality to $[0,1]^d$:

$$W_2^2(\mu \to \mu^*) \lesssim \int_{[0,1]^d} \left|\nabla^- f\right|^2 \mathrm{d}x .$$

4. Use *Kantorovich duality* to go from Wasserstein to L^p distance:

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Step 1: The directed heat equation

Goal: find a robust approach to the classical Poincaré inequality, such that "toggling" one aspect of this approach yields the **directed** version.

Main idea: the classical Poincaré inequality characterizes the convergence of the heat equation toward equilibrium.

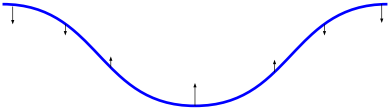
The heat equation

Let $x \in [0,1]^d$ denote space and $t \geq 0$ time. For $u = u(t,x)$, the heat equation is

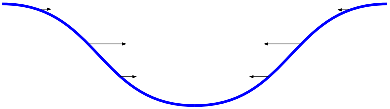
$$\partial_t u = \Delta u\,,$$

where $\Delta = \sum_i \partial_i \partial_i$ is the Laplacian operator. In one dimension,

$$\partial_t u = \partial_x \underbrace{\partial_x u}_{\text{“flux”}}\ .$$



Value view



Particle view

Exponential convergence

Variance decay: assuming u has mean zero,

$$\begin{aligned}
\partial_t \mathrm{Var}\left[u(t)\right] &= \partial_t \int u(t)^2 \,\mathrm{d}x \\
&= \int \partial_t u(t)^2 \,\mathrm{d}x \\
&= 2 \int u(t) \Delta u(t) \,\mathrm{d}x && \text{(Heat equation)} \\
&= -2 \int \nabla u(t) \cdot \nabla u(t) && \text{(Integration by parts)} \\
&= -2\mathbb{E}\left[\|\nabla u(t)\|_2^2\right] .
\end{aligned}$$

So $\partial_t \mathrm{Var}\left[u(t)\right] \le -C\mathrm{Var}\left[u(t)\right]$ iff $\mathrm{dist}_2^{\mathrm{const}}(u(t))^2 = \mathrm{Var}\left[u(t)\right] \le \frac{2}{C}\mathbb{E}\left[\|\nabla u(t)\|_2^2\right]$.

Exponential decay of variance is equivalent to the Poincaré inequality.

Dirichlet energy

Looking ahead at the **directed** case: analyze decay of $\mathrm{dist}_2^{\mathrm{mono}}(f)^2$? Maybe, but does not seem to lead to a proof.

Another relevant quantity: the Dirichlet energy

$$\mathcal{E}(f) = \frac{1}{2} \int (\partial_x f)^2 \, \mathrm{d}x \, .$$

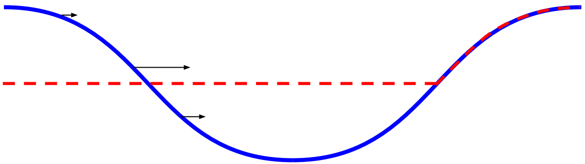
Intuition: measure the **local violations** of the "constant" property.

By similar calculation, $\mathcal{E}(u(t))$ also enjoys exponential decay under the heat equation.

Directed heat equation

Sticking to one dimension for now, we study the directed heat equation

$$\partial_t u = \partial_x \partial_x^- u(t) \,.$$



Particle view, directed version

"Should" converge to a *monotone* limit $\implies$ learn about distance to monotonicity?

Analyze via the directed Dirichlet energy

$$\mathcal{E}^-(f) = \frac{1}{2} \int (\partial_x^- f)^2 \, \mathrm{d}x \,.$$

PDE results

Using $\mathcal{E}^-$, bring in the theory of gradient flows and maximal monotone operators.

- The directed heat equation has a solution $u \mapsto P_t u$ (directed heat semigroup).
- $\mathcal{E}^-(P_t u)$ decays exponentially in time.
- The solution converges to a **monotone** limit $P_\infty u$ as $t \to \infty$.
- Nice analytic properties such as
  - Nonexpansiveness: $\|P_t u - P_t v\|_{L^2} \leq \|u - v\|_{L^2}$.
  - Order preservation: $u \leq v \implies P_t u \leq P_t v$.
  - Regularity preservation: u "regular" $\implies P_t u$ "regular" (concretely: Lipschitz, Sobolev class H^1).
- Main idea: canonical decomposition $u = u\!\uparrow + u\!\downarrow$.
  - $u\!\downarrow$ is "well-behaved" ($\approx$ differentiable), $u\!\uparrow$ can be "wild" (e.g. jump up).

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Directed Poincaré inequality – proof overview

Directed Poincaré inequality:

$$\mathrm{dist}_2^{\mathrm{mono}}(f)^2 \lesssim \mathbb{E}\left[\|\nabla^- f\|_2^2\right] .$$

Main ideas:

1. Introduce and study a 1D dynamical process, the directed heat equation.
2. Use it to to prove the 1D *transport-energy inequality*

$$W_2^2(\mu, \mu_\infty) \lesssim \int_{[0,1]} (\partial_x^- u)^2 \, \mathrm{d}x .$$

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Optimal transport

Intimate connection between PDEs and optimal transport. For probability measures ϱ_0, ϱ_1, the squared Wasserstein distance

$$W_2^2(\varrho_0, \varrho_1) = \text{minimum cost of moving mass from } \varrho_0 \text{ to } \varrho_1 \,,$$

if moving mass from x to y costs $|x - y|^2$.

Connection to PDEs via the Benamou-Brenier formula:

$$W_2^2(\varrho_0, \varrho_1) = \min \left\{ \int_0^1 \|v_t\|_{L^2(\varrho_t)}^2 \,\mathrm{d}t : v_t \text{ velocity field taking } \varrho_0 \text{ to } \varrho_1 \text{ from time 0 to 1} \right\} .$$

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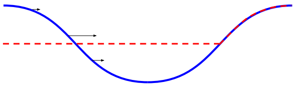


Directed heat equation $\partial_t u = \partial_x \partial_x^- u$

Suggests connection to the directed Dirichlet energy, $\mathcal{E}^-(f) = \frac{1}{2}\int (\partial_x^- f)^2\,\mathrm{d}x$.

"velocity $v_t \approx$ momentum $\approx$ flux $\approx \partial_x^- f$"

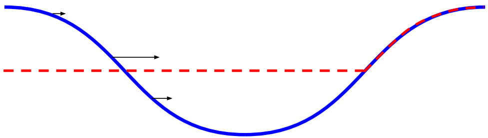
Step 2 – 1D Transport-energy inequality

Exploit the exponential decay of $\mathcal{E}^-$, via Benamou-Brenier, to conclude

Theorem (Transport-energy inequality in one dimension)

There exists a constant $C > 0$ such that the following holds. Let $u \in \mathcal{U}$ be positive, bounded away from zero, and satisfy $\int_{(0,1)} u \, \mathrm{d}x = 1$. Define the measures $\mathrm{d}\mu := u \, \mathrm{d}x$ and $\mathrm{d}\mu_\infty := (P_\infty u) \, \mathrm{d}x$. Then

$$W_2^2(\mu, \mu_\infty) \leq \frac{C}{\inf u} \mathcal{E}^-(u) \, .$$



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Step 3 – Tensorizing the transport-energy inequality

How to **tensorize** into a multidimensional result? Key: move along one coordinate at a time, compose the costs $|x-y|^2$ via the Pythagorean theorem.

1. Make all rows monotone, particles move horizontally.
2. Make all columns monotone, particles move vertically.
3. ...

Theorem (Transport-energy inequality)

There exists a universal constant $C > 0$ such that the following holds. Let $a \in (0,1)$, and let $f \in \mathrm{Lip}$ satisfy $1-a \le f \le 1+a$ and $\int_{[0,1]^d} f \,\mathrm{d}x = 1$. Define the probability measures $\mathrm{d}\mu := f\,\mathrm{d}x$ and $\mathrm{d}\mu^* := f^*\,\mathrm{d}x$ on $[0,1]^d$. Then

$$W_2^2(\mu \to \mu^*) \le \frac{C(1+a)^2}{(1-a)^3} \int_{[0,1]^d} \left|\nabla^- f\right|^2 \mathrm{d}x \,.$$

Note above directed Wasserstein distance!

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Step 4 – From Wasserstein to L^p

Classically: close connections between transport-energy inequalities, Poincaré inequalities, Talagrand concentration inequalities, and log-Sobolev inequalities [Villani'09]. In particular [Liu'19] gives equivalence between:

1. **Transport-energy inequality**:

$$W_2^2(\mu, \mu_{\mathrm{unif}}) \lesssim \mathbb{E}\left[\|\nabla f\|_2^2\right] , \qquad \text{where } \mathrm{d}\mu := f\,\mathrm{d}x = f\,\mathrm{d}\mu_{\mathrm{unif}} .$$

2. **Poincaré inequality**:

$$\mathrm{dist}_2^{\mathrm{const}}(f)^2 \lesssim \mathbb{E}\left[\|\nabla f\|_2^2\right] .$$

We show a directed *implication*. If the **directed transport energy inequality** holds:

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☺

Main ingredient: Kantorovich duality

Conclusion and open questions

- Main message: convergence properties of a PDE are the principle underlying **classical** and **directed** Poincaré inequalities.
- We extend the connection between directed isoperimetry and monotonicity testing, which proves to be robust to the choice of continuous/discrete setting.
- The $\widetilde{O}(\sqrt{d}M^2/\varepsilon^2)$ tester helps unite the continuous and discrete landscapes of monotonicity testing.

Questions

- Our tester is a continuous analogue of the path tester of [KMS'18]. Is there a *formal connection* between the discrete and continuous cases?
- Lower bounds for general testers in the continuous setting?
- Other applications of the dynamical approach (PDEs, optimal transport theory) to property testing? Maybe other questions in TCS?