CS886: Influence diagrams

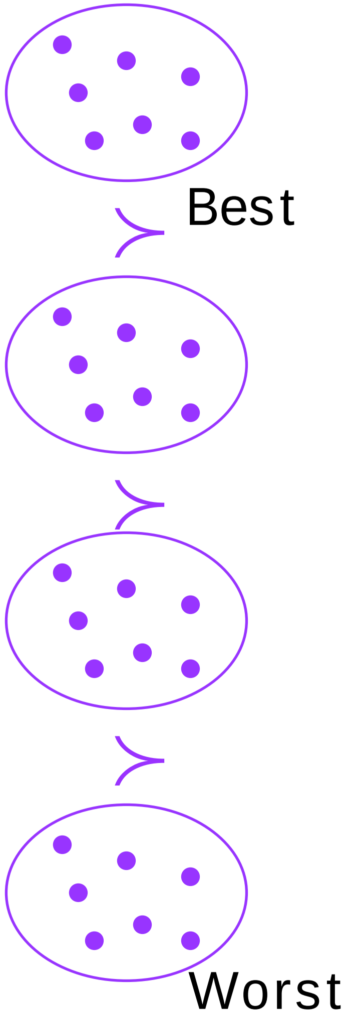
- January 25^{th}
- Intro to utility theory and decision theory
- Influence diagrams (or decision networks)
- Decision making under uncertainty

Preference Orderings

- A *preference ordering* $\succcurlyeq$ is a ranking of all possible states of affairs (worlds) S
  - these could be outcomes of actions, truth assts, states in a search problem, etc.
  - s $\succcurlyeq$ t: means that state s is *at least as good as* t
  - s $\succ$ t: means that state s is *strictly preferred to* t
- We insist that $\succcurlyeq$ is
  - reflexive: i.e., s $\succcurlyeq$ s for all states s
  - transitive: i.e., if s $\succcurlyeq$ t and t $\succcurlyeq$ w, then s $\succcurlyeq$ w
  - connected: for all states s,t, either s $\succcurlyeq$ t or t $\succcurlyeq$ s

Why Impose These Conditions?

- Structure of preference ordering imposes certain “rationality requirements” (it is a weak ordering)
- E.g., why transitivity?
  - Suppose you (strictly) prefer coffee to tea, tea to OJ, OJ to coffee
  - If you prefer X to Y, you’ll trade me Y plus $1 for X
  - I can construct a “money pump” and extract arbitrary amounts of money from you



Utilities

- Rather than just ranking outcomes, we must quantify our degree of preference
  - e.g., how much more important is chc than ~mess
- A *utility function* $U:S \rightarrow \mathbb{R}$ associates a real-valued *utility* with each outcome.
  - U(s) measures your *degree* of preference for s
- Note: U induces a preference ordering $\succcurlyeq_U$ over S defined as: $s \succcurlyeq_U t$ iff $U(s) \geq U(t)$
  - obviously $\succcurlyeq_U$ will be reflexive, transitive, connected

Expected Utility

- Under conditions of uncertainty, each decision d induces a distribution $\Pr_d$ over possible outcomes
  - $\Pr_d(s)$ is probability of outcome s under decision d
- The *expected utility* of decision d is defined

$$EU(d) = \sum_{s \in S} \Pr_d(s) U(s)$$

The MEU Principle

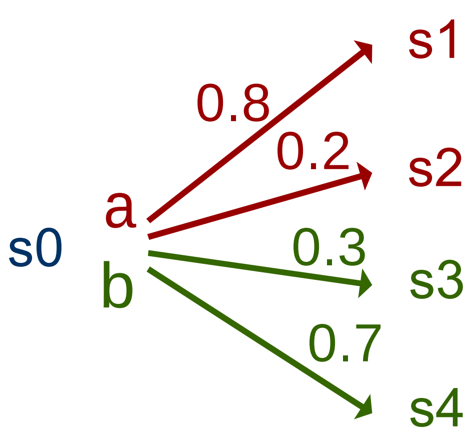
- The *principle of maximum expected utility (MEU)* states that the optimal decision under conditions of uncertainty is that with the greatest expected utility.
- In our example
  - if my utility function is the first one, my robot should get coffee
  - if your utility function is the second one, your robot should do nothing

Decision Problems: Uncertainty

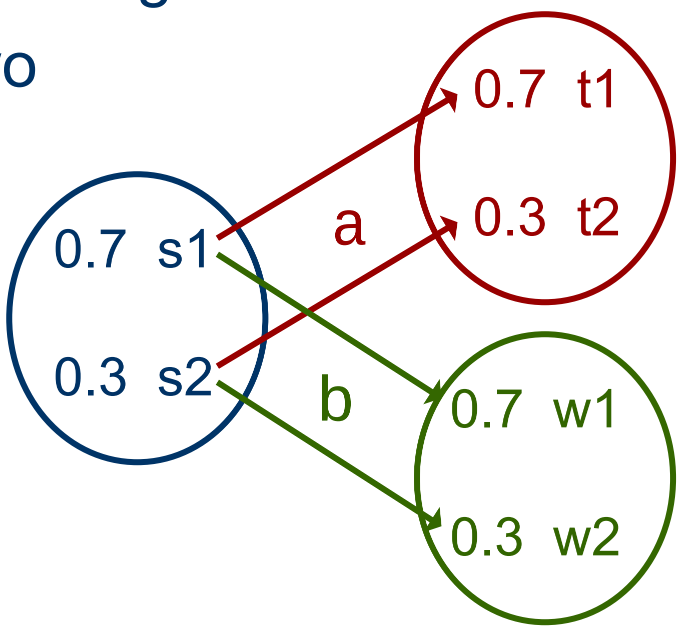
- A *decision problem under uncertainty* is:
  - a set of *decisions* D
  - a set of *outcomes* or states S
  - an *outcome function* $Pr : D \rightarrow \Delta(S)$
    - $\Delta(S)$ is the set of distributions over S (e.g., Pr_d)
  - a utility function U over S
- A *solution* to a decision problem under uncertainty is any $d^* \in D$ such that $EU(d^*) \succcurlyeq EU(d)$ for all $d \in D$
- Again, for single-shot problems, this is trivial

Expected Utility: Notes

- Note that this viewpoint accounts for both:
  - uncertainty in action outcomes
  - uncertainty in state of knowledge
  - any combination of the two



Stochastic actions



Uncertain knowledge

Expected Utility: Notes

- Why MEU? Where do utilities come from?
  - underlying foundations of utility theory tightly couple utility with action/choice
  - a utility function can be determined by asking someone about their preferences for actions in specific scenarios (or “lotteries” over outcomes)
- Utility functions needn’t be unique
  - if I multiply U by a positive constant, all decisions have same relative utility
  - if I add a constant to U, same thing
  - *U is unique up to positive affine transformation*

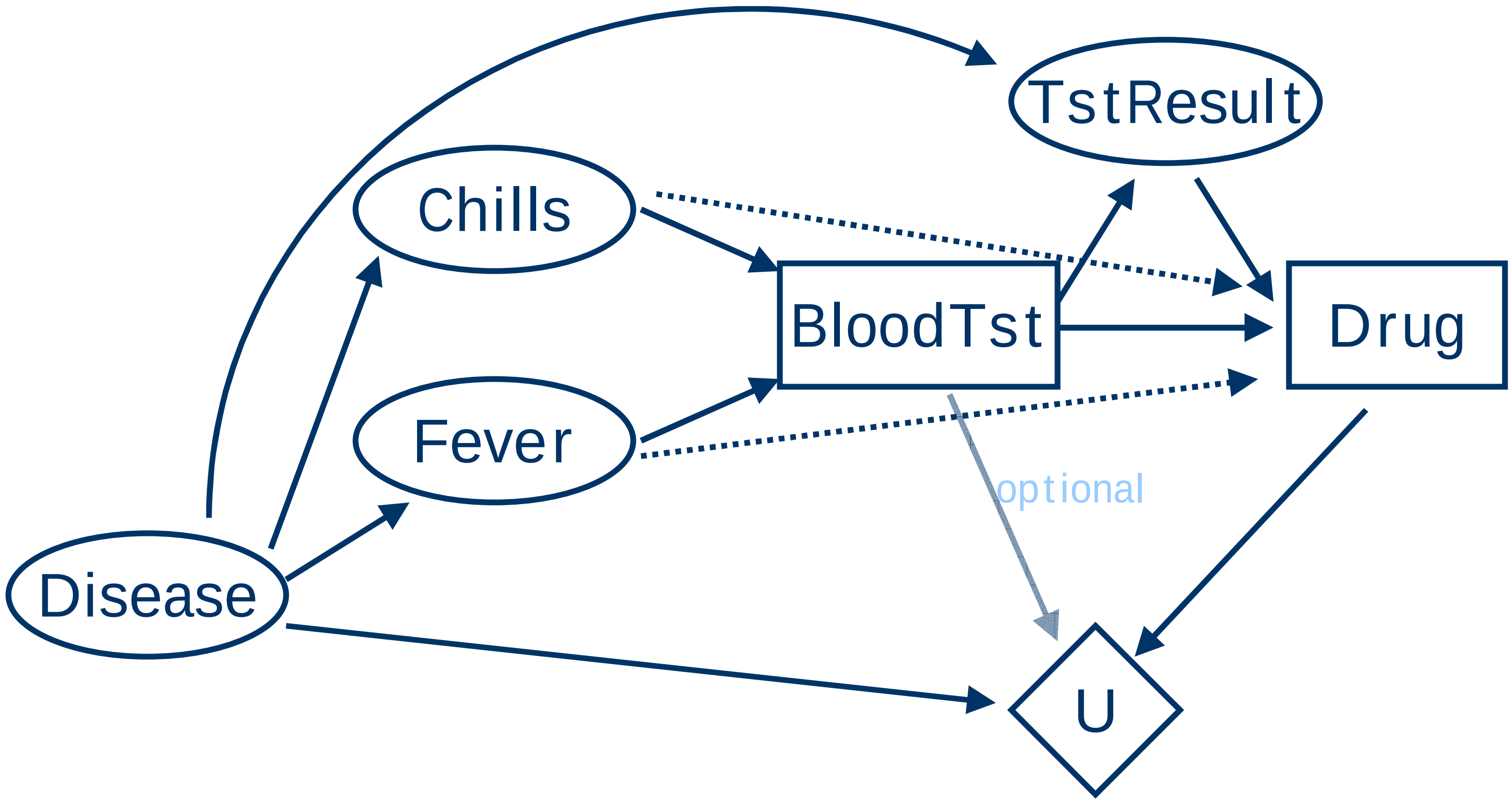
So What are the Complications?

- Outcome space is large
  - like all of our problems, states spaces can be huge
  - don't want to spell out distributions like Pr_d explicitly
  - Soln: Bayes nets (or related: *influence diagrams*)
- Decision space is large
  - usually our decisions are not one-shot actions
  - rather they involve sequential choices (like plans)
  - if we treat each plan as a distinct decision, decision space is too large to handle directly
  - Soln: use dynamic programming methods to *construct* optimal plans (actually generalizations of plans, called policies… like in game trees)

Decision Networks

- *Decision networks* (more commonly known as *influence diagrams*) provide a way of representing sequential decision problems
  - basic idea: represent the variables in the problem as you would in a BN
  - add decision variables – variables that you “control”
  - add utility variables – how good different states are

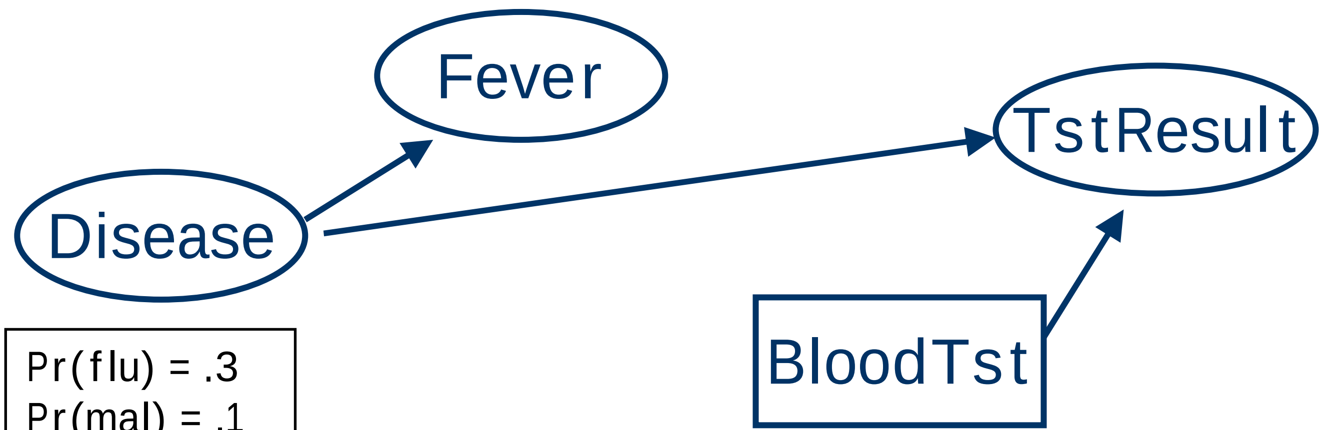
Sample Decision Network



Decision Networks: Chance Nodes

- **Chance nodes**
  - random variables, denoted by circles
  - as in a BN, probabilistic dependence on parents

Pr(f|flu) = .5
Pr(f|mal) = .3
Pr(f|none) = .05

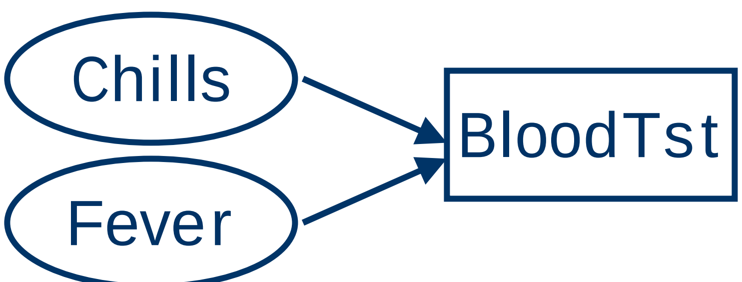


Pr(flu) = .3
Pr(mal) = .1
Pr(none) = .6

Pr(pos|flu,bt) = .2
Pr(neg|flu,bt) = .8
Pr(null|flu,bt) = 0
Pr(pos|mal,bt) = .9
Pr(neg|mal,bt) = .1
Pr(null|mal,bt) = 0
Pr(pos|no,bt) = .1
Pr(neg|no,bt) = .9
Pr(null|no,bt) = 0
Pr(pos|D,~bt) = 0
Pr(neg|D,~bt) = 0
Pr(null|D,~bt) = 1

Decision Networks: Decision Nodes

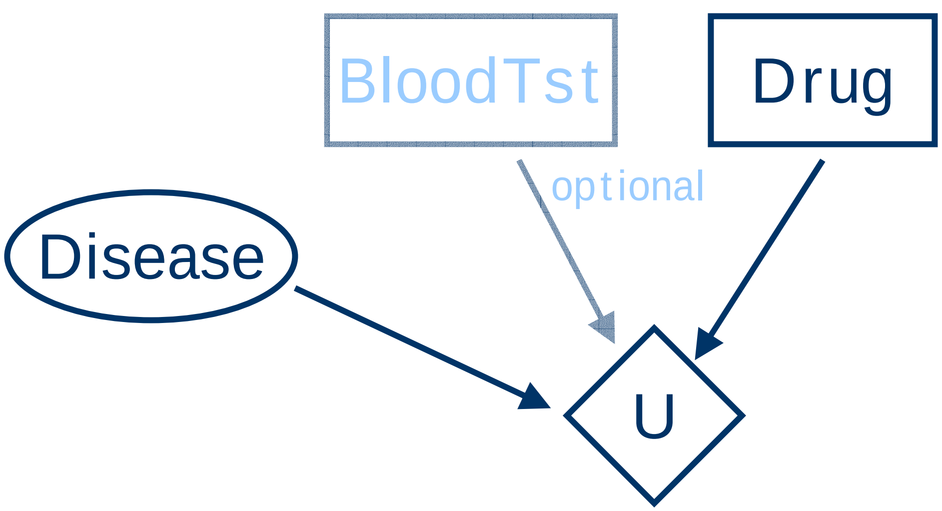
- **Decision nodes**
  - variables decision maker sets, denoted by squares
  - parents reflect *information available* at time decision is to be made
- In example decision node: the actual values of Ch and Fev will be observed before the decision to take test must be made
  - agent can make *different decisions* for each instantiation of parents (i.e., policies)



BT ϵ {bt, ~bt}

Decision Networks: Value Node

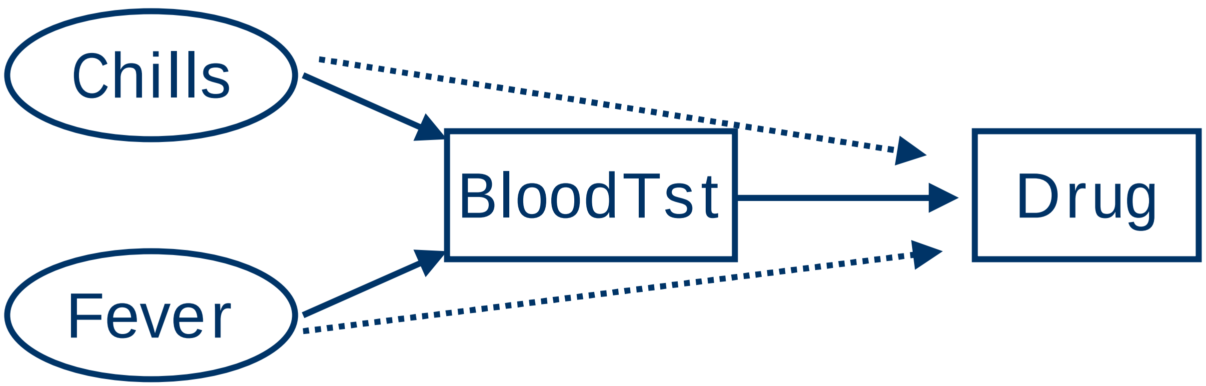
- **Value node**
  - specifies utility of a state, denoted by a diamond
  - utility depends *only on state of parents* of value node
  - generally: only one value node in a decision network
- Utility depends only on disease and drug



U(fludrug, flu) = 20
U(fludrug, mal) = -300
U(fludrug, none) = -5
U(maldrug, flu) = -30
U(maldrug, mal) = 10
U(maldrug, none) = -20
U(no drug, flu) = -10
U(no drug, mal) = -285
U(no drug, none) = 30

Decision Networks: Assumptions

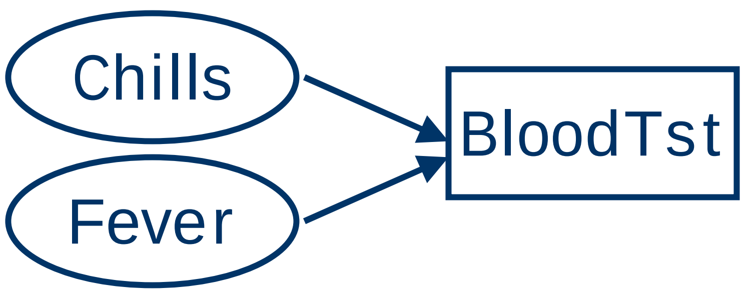
- Decision nodes are totally ordered
  - decision variables $D_1, D_2, \ldots, D_n$
  - decisions are made in sequence
  - e.g., BloodTst (yes,no) decided before Drug (fd,md,no)
- *No-forgetting property*
  - any information available when decision D_i is made is available when decision D_j is made (for $i < j$)
  - thus all parents of D_i are parents of D_j



Dashed arcs ensure the no-forgetting property

Policies

- Let $Par(D_i)$ be the parents of decision node D_i
  - $Dom(Par(D_i))$ is the set of assignments to parents
- A policy δ is a set of mappings δ_i, one for each decision node D_i
  - $\delta_i : Dom(Par(D_i)) \rightarrow Dom(D_i)$
  - δ_i associates a decision with each parent asst for D_i
- For example, a policy for BT might be:
  - $\delta_{BT}(c,f) = bt$
  - $\delta_{BT}(c,\sim f) = \sim bt$
  - $\delta_{BT}(\sim c,f) = bt$
  - $\delta_{BT}(\sim c,\sim f) = \sim bt$



Value of a Policy

- *Value of a policy δ* is the expected utility given that decision nodes are executed according to δ
- Given asst **x** to the set **X** of all chance variables, let $\delta(\mathbf{x})$ denote the asst to decision variables dictated by δ
  - e.g., asst to D_1 determined by it's parents' asst in **x**
  - e.g., asst to D_2 determined by it's parents' asst in **x** along with whatever was assigned to D_1
  - etc.
- Value of δ :

$$EU(\delta) = \Sigma_{\mathbf{X}} P(\mathbf{X}, \delta(\mathbf{X}))\, U(\mathbf{X}, \delta(\mathbf{X}))$$

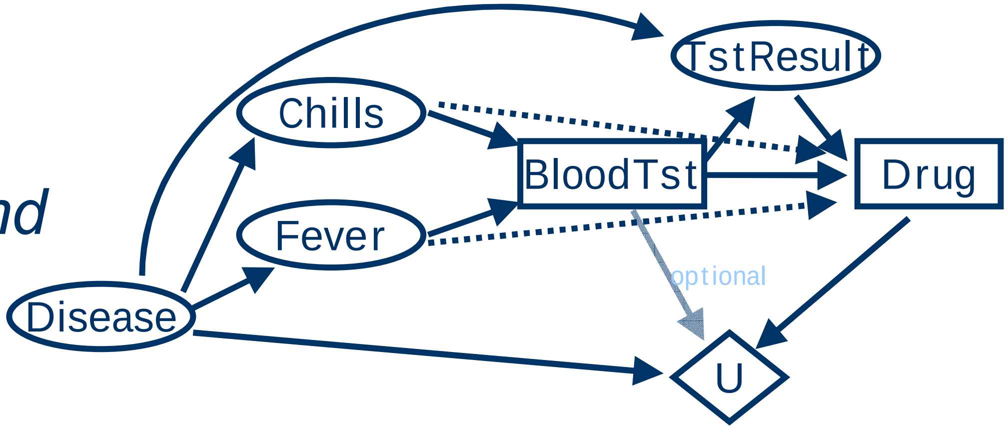
Optimal Policies

- An *optimal policy* is a policy δ^* such that

  $EU(\delta^*) \geq EU(\delta)$ for all policies δ
- We can use the dynamic programming principle yet again to avoid enumerating all policies
- We can also use the structure of the decision network to use variable elimination to aid in the computation

Computing the Best Policy

- We can work backwards as follows
- First compute optimal policy for Drug (last dec'n)
  - for each asst to parents (C,F,BT,TR) and for each decision value (D = md,fd,none), *compute the expected value* of choosing that value of D
  - set policy choice for each value of parents to be the value of D that has max value
  - eg: $\delta_D(c,f,bt,pos) = md$

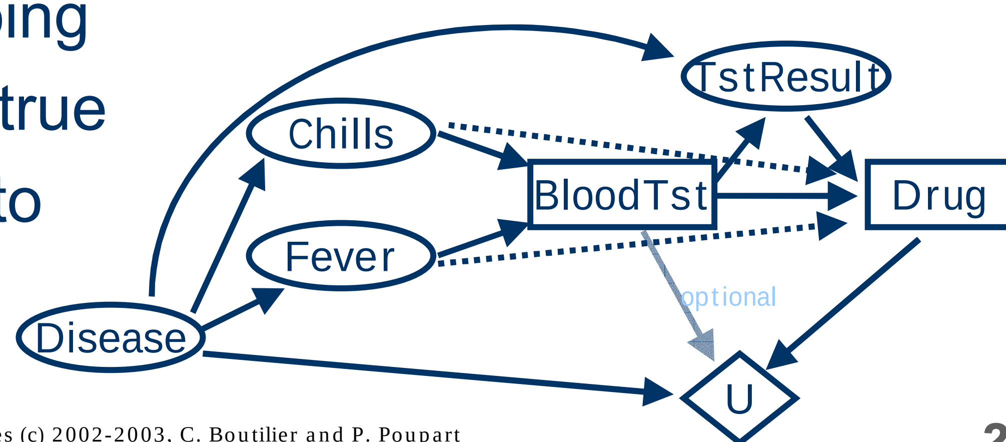


Computing the Best Policy

- Next compute policy for BT given policy $\delta_D(C,F,BT,TR)$ just determined for Drug
  - since $\delta_D(C,F,BT,TR)$ is fixed, we can treat Drug as a normal random variable with deterministic probabilities
  - i.e., for any instantiation of parents, value of Drug is fixed by policy δ_D
  - this means we can solve for optimal policy for BT just as before
  - only uninstantiated vars are random vars (once we fix *its* parents)

Computing the Best Policy

- How do we compute these expected values?
  - suppose we have asst <*c,f,bt,pos*> to parents of *Drug*
  - we want to compute EU of deciding to set *Drug* = *md*
  - we can run variable elimination!
- Treat *C,F,BT,TR,Dr* as evidence
  - this reduces factors (e.g., *U* restricted to *bt,md*: depends on *Dis*)
  - eliminate remaining variables (e.g., only *Disease* left)
  - left with factor: $\mathbf{U() = \Sigma_{Dis}\ P(Dis|\ c,f,bt,pos,md)U(Dis)}$
- We now know EU of doing *Dr=md* when *c,f,bt,pos* true
- Can do same for *fd,no* to decide which is best

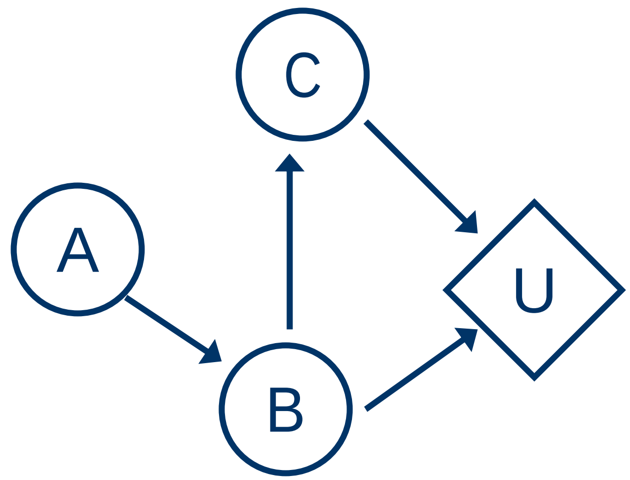


Computing Expected Utilities

- The preceding illustrates a general phenomenon
  - computing expected utilities with BNs is quite easy
  - utility nodes are just factors that can be dealt with using variable elimination

$$EU = \Sigma_{A,B,C} P(A,B,C)\ U(B,C)$$
$$= \Sigma_{A,B,C} P(C|B)\ P(B|A)\ P(A)\ U(B,C)$$

- Just eliminate variables in the usual way



Optimizing Policies: Key Points

- If a decision node D has no decisions that follow it, we can find its policy by instantiating each of its parents and computing the expected utility of each decision for each parent instantiation
  - no-forgetting means that all other decisions are instantiated (they must be parents)
  - its easy to compute the expected utility using VE
  - the number of computations is quite large: we run expected utility calculations (VE) for each parent instantiation together with each possible decision D might allow
  - policy: choose max decision for each parent instant’n

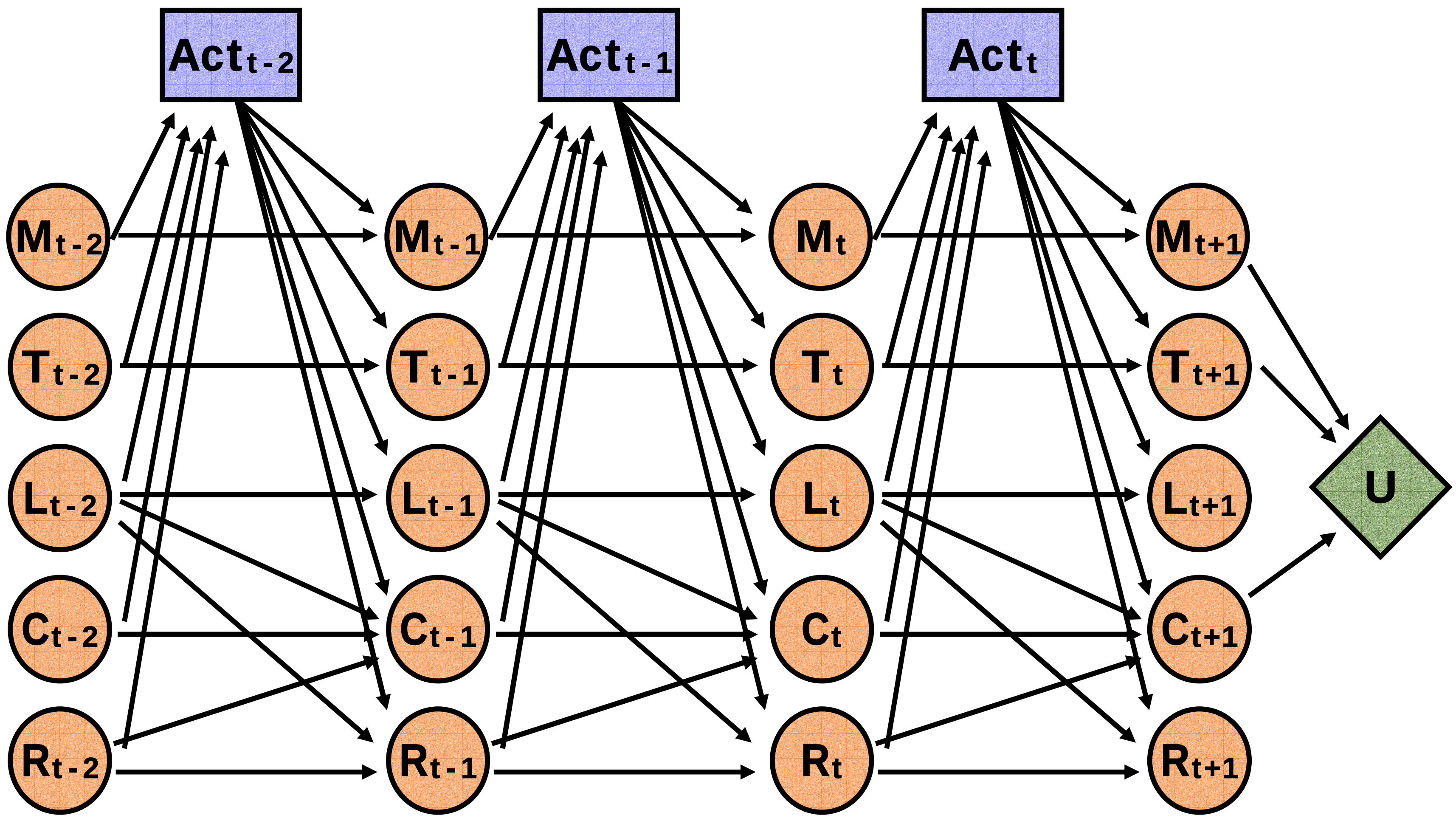
Optimizing Policies: Key Points

- When a decision D node is optimized, it can be treated as a random variable
  - for each instantiation of its parents we now know what value the decision should take
  - just treat policy as a new CPT: for a given parent instantiation **x**, D gets $\delta(\mathbf{x})$ with probability 1(all other decisions get probability zero)
- If we optimize from last decision to first, at each point we can optimize a specific decision by (a bunch of) simple VE calculations
  - it's successor decisions (optimized) are just normal nodes in the BNs (with CPTs)

Decision Network Notes

- Decision networks commonly used by decision analysts to help structure decision problems
- Much work put into computationally effective techniques to solve these
  - common trick: replace the decision nodes with random variables at outset and solve a plain Bayes net (a subtle but useful transformation)
- Complexity much greater than BN inference
  - we need to solve a number of BN inference problems
  - one BN problem for each setting of decision node parents and decision node value

DBN-Decision Nets for Planning



DBN Decision Networks

- In example on previous slide:
  - we assume the state (of the variables at any stage) is fully observable
    - hence all time t vars point to time t decision
  - this means the state at time t d-separates the decision at time t-1 from the decision at time t-2
  - so we ignore “no-forgetting” arcs between decisions
    - once you *know* the state at time t, what you *did* at time t-1 to get there is irrelevant to the decision at time t-1
- If the state were not fully observable, we could not ignore the “no-forgetting” arcs

A Detailed Decision Net Example

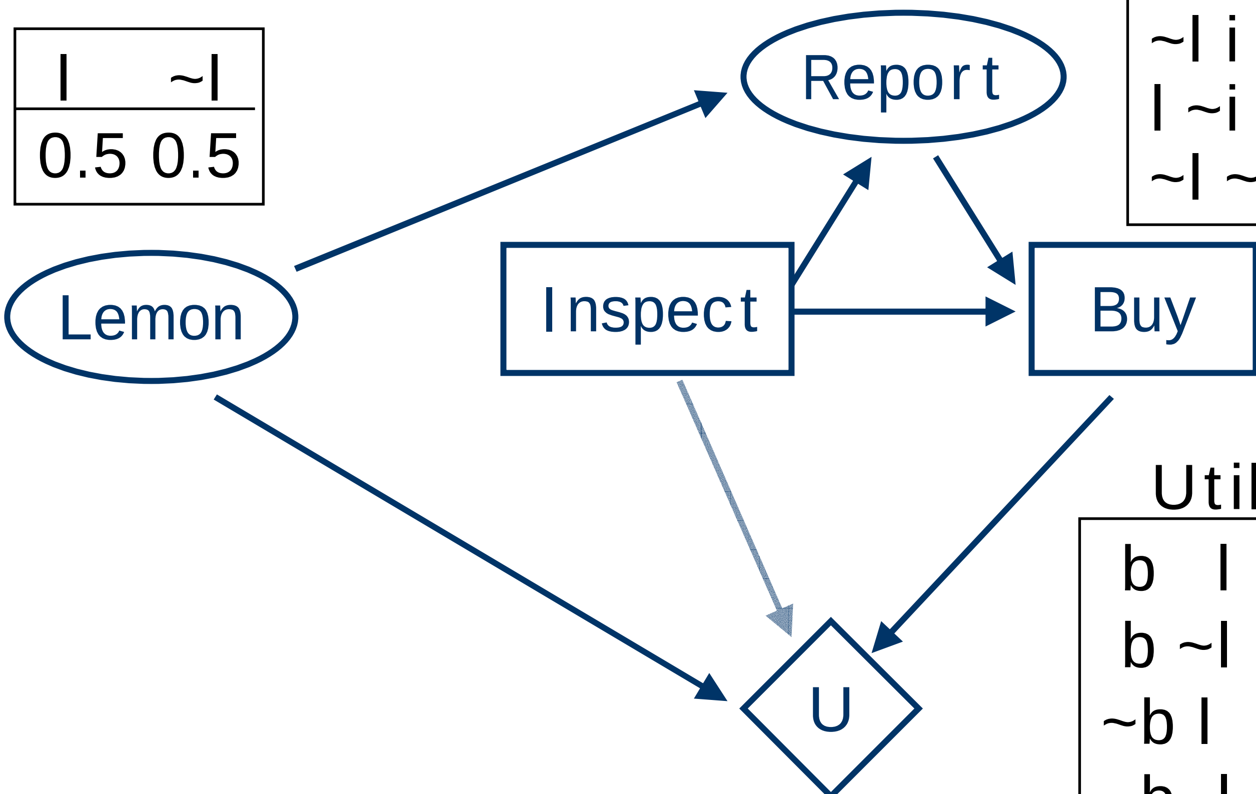
- Setting: you want to buy a used car, but there's a good chance it is a "lemon" (i.e., prone to breakdown). Before deciding to buy it, you can take it to a mechanic for inspection. S/he will give you a report on the car, labelling it either "good" or "bad". A good report is positively correlated with the car being sound, while a bad report is positively correlated with the car being a lemon.
- The report costs $50 however. So you could risk it, and buy the car without the report.
- Owning a sound car is better than having no car, which is better than owning a lemon.

Car Buyer's Network

Rep: good,bad,none

	g	b	n
l i	0.2	0.8	0
~l i	0.9	0.1	0
l ~i	0	0	1
~l ~i	0	0	1

l	~l
0.5	0.5



Utility

b l	-600
b ~l	1000
~b l	-300
~b~l	-300

-50 if inspect

Evaluate Last Decision: Buy (1)

- $EU(B|I,R) = \Sigma_L \; P(L|I,R,B) \; U(L,B)$
- I = i, R = g:
  - EU(buy) = P(l|i, g) U(l,buy) + P(~l|i, g) U(~l,buy) - 50
    = .18*-600 + .82*1000 - 50 = 662
  - EU(~buy) = P(l|i, g) U(l,~buy) + P(~l|i, g) U(~l,~buy) - 50
    = -300 - 50 = -350 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(i,g) = buy*

Evaluate Last Decision: Buy (2)

- l = i, R = b:
  - EU(buy) = P(l|i, b) U(l,buy) + P(~l|i, b) U(~l,buy) - 50
    = .89*-600 + .11*1000 - 50 = -474
  - EU(~buy) = P(l|i, b) U(l,~buy) + P(~l|i, b) U(~l,~buy) - 50
    = -300 - 50 = -350 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(i,b) = ~buy*

Evaluate Last Decision: Buy (3)

- I = ~i, R = n (note: no inspection cost subtracted)
  - EU(buy) = P(l|~i, n) U(l,buy) + P(~l|~i, n) U(~l,buy)
    = .5*-600 + .5*1000 = 200
  - EU(~buy) = P(l|~i, n) U(l,~buy) + P(~l|~i, n) U(~l,~buy)
    = -300 - 50 = -350 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(~i,n) = buy*
- So optimal policy for Buy is:
  - δ_{Buy} *(i,g) = buy* ; δ_{Buy} *(i,b) = ~buy* ; δ_{Buy} *(~i,n) = buy*
- Note: we don't bother computing policy for (i,~n), (~i, g), or (~i, b), since these occur with probability 0

Evaluate First Decision: Inspect

- $EU(I) = \Sigma_{L,R} P(L,R|I)\, U(L, \boldsymbol{\delta_{Buy}(I,R)})$
  - where $P(R,L|I) = P(R|L,I)P(L|I)$
  - $EU(i) = .1*-600 + .4*-300 + .45*1000 + .05*-300 - 50$
    $= 237.5 – 50 = 187.5$
  - $EU(\sim i) = P(l|\sim i, n)\, U(l,buy) + P(\sim l|\sim i, n)\, U(\sim l,buy)$
    $= .5*-600 + .5*1000 = 200$
  - So optimal $\delta_{Inspect}$ *(~i) = buy*

	P(R,L | I)	$\boldsymbol{\delta_{Buy}}$	U(L, $\boldsymbol{\delta_{Buy}}$)
g,l	0.1	buy	-600 - 50 = -650
b,l	0.4	~buy	-300 - 50 = -350
g,~l	0.45	buy	1000 - 50 = 950
b,~l	0.05	~buy	-300 - 50 = -350

Value of Information

- So optimal policy is: don’t inspect, buy the car
  - EU = 200
  - Notice that the EU of inspecting the car, then buying it iff you get a good report, is 237.5 less the cost of the inspection (50). So inspection not worth the improvement in EU.
  - But suppose inspection cost $25: then it would be worth it (EU = 237.5 – 25 = 212.5 > EU(~i))
  - The *expected value of information* associated with inspection is 37.5 (it improves expected utility by this amount ignoring cost of inspection). How? Gives opportunity to change decision (~buy if bad).
  - You should be willing to pay up to $37.5 for the report