Lecture 2a: Convergence Properties
CS885 Reinforcement Learning

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Complementary readings: [SutBar] Sec. 4.1, 4.4, [Sze] Sec. 2.2, 2.3, [Put] Sec. 6.1-6.3, [SigBuf] Chap. 1

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Outline

- Convergence properties of
  - Policy evaluation
  - Value iteration

Recap: Value Iteration

- Matrix form:

  R^a: $|S| \times 1$ column vector of rewards for a
  V_n^*: $|S| \times 1$ column vector of state values
  T^a: $|S| \times |S|$ matrix of transition prob. for a

valueIteration(MDP)

$V_0^* \leftarrow \max_a R^a$

For $t = 1$ to h do

$V_n^* \leftarrow \max_a R^a + \gamma T^a V_{n-1}^*$

Return V^*

Infinite Horizon

- Let $h \to \infty$
- Then $V_h^\pi \to V_\infty^\pi$ and $V_{h-1}^\pi \to V_\infty^\pi$

- **Policy evaluation:**

$$V_\infty^\pi(s) = R\big(s, \pi_\infty(s)\big) + \gamma \sum_{s'} \Pr(s'|s, \pi_\infty(s))\, V_\infty^\pi(s') \;\; \forall s$$

- **Bellman's equation:**

$$V_\infty^*(s) = \max_a R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_\infty^*(s')$$

Policy evaluation

- Linear system of equations

$$V_\infty^\pi(s) = R\big(s, \pi_\infty(s)\big) + \gamma \sum_{s'} \Pr(s'|s, \pi_\infty(s))\, V_\infty^\pi(s') \;\; \forall s$$

- Matrix form:

  R: $|S| \times 1$ column vector of sate rewards for π

  V: $|S| \times 1$ column vector of state values for π

  T: $|S| \times |S|$ matrix of transition probabilities for π

$$V = R + \gamma T V$$

Solving linear equations

- Linear system: $V = R + \gamma TV$

- Gaussian elimination: $(I - \gamma T)V = R$
- Compute inverse: $V = (I - \gamma T)^{-1}R$
- Iterative methods
  - Value iteration (a.k.a. Richardson iteration)
  - Repeat $V \leftarrow R + \gamma TV$

Contraction

- Let $H(V) \stackrel{\text{def}}{=} R + \gamma T V$ be the policy evaluation operator
- **Lemma 1**: H is a contraction mapping.

$$\left|\left|H(\tilde{V}) - H(V)\right|\right|_\infty \leq \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$$

- Proof $\left|\left|H(\tilde{V}) - H(V)\right|\right|_\infty = \left|\left|R + \gamma T\tilde{V} - R - \gamma TV\right|\right|_\infty$ (by definition)

$$= \left|\left|\gamma T(\tilde{V} - V)\right|\right|_\infty \quad \text{(simplification)}$$

$$\leq \gamma ||T||_\infty \left|\left|\tilde{V} - V\right|\right|_\infty \quad \text{(since } ||AB|| \leq ||A||\,||B|| \text{)}$$

$$= \gamma \left|\left|\tilde{V} - V\right|\right|_\infty \quad \text{(since } \max_s \textstyle\sum_{s'} T(s, s') = 1\text{)}$$

Convergence

- **Theorem 2**: Policy evaluation converges to V^{π} for any initial estimate V

$$\lim_{n\to\infty} H^{(n)}(V) = V^{\pi} \quad \forall V$$

- Proof
  - By definition $\mathrm{V}^{\pi} = H^{(\infty)}(0)$, but policy evaluation computes $H^{(\infty)}(V)$ for any initial V
  - By Lemma 1, $\left|\left|H^{(n)}(V) - H^{(n)}(\tilde{V})\right|\right|_{\infty} \leq \gamma^{n}\left|\left|V - \tilde{V}\right|\right|_{\infty}$
  - Hence, when $n \to \infty$, then $\left|\left|H^{(n)}(V) - H^{(n)}(0)\right|\right|_{\infty} \to 0$ and $H^{(\infty)}(V) = V^{\pi} \;\; \forall V$

Approximate Policy Evaluation

- In practice, we can't perform an infinite number of iterations.
- Suppose that we perform value iteration for n steps and $\left|\left|H^{(n)}(V) - H^{(n-1)}(V)\right|\right|_\infty = \epsilon$, how far is $H^{(n)}(V)$ from V^π?

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Approximate Policy Evaluation

- **Theorem 3**: If $\left|\left|H^{(n)}(V) - H^{(n-1)}(V)\right|\right|_\infty \leq \epsilon$ then

$$\left|\left|V^\pi - H^{(n)}(V)\right|\right|_\infty \leq \frac{\epsilon}{1-\gamma}$$

- Proof $\left|\left|V^\pi - H^{(n)}(V)\right|\right|_\infty = \left|\left|H^{(\infty)}(V) - H^{(n)}(V)\right|\right|_\infty$ (by Theorem 2)

$= \left|\left|\sum_{t=1}^{\infty} H^{(t+n)}(V) - H^{(t+n-1)}(V)\right|\right|_\infty$

$\leq \sum_{t=1}^{\infty} \left|\left|H^{(t+n)}(V) - H^{(t+n-1)}(V)\right|\right|_\infty$ $(||A+B|| \leq ||A|| + ||B||)$

$\leq \sum_{t=1}^{\infty} \gamma^t \epsilon = \frac{\epsilon}{1-\gamma}$ (by Lemma 1)

Optimal Value Function

- Non-linear system of equations

$$V_\infty^*(s) = \max_a R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_\infty^*(s') \;\; \forall s$$

- Matrix form:

  R^a: $|S| \times 1$ column vector of rewards for a

  V^*: $|S| \times 1$ column vector of optimal values

  T^a: $|S| \times |S|$ matrix of transition prob for a

$$V^* = \max_a R^a + \gamma T^a V^*$$

Contraction

- Let $H^*(V) \stackrel{\text{def}}{=} \max_a R^a + \gamma T^a V$ be the operator in value iteration
- **Lemma 4**: H^* is a contraction mapping. Then $\left|\left|H^*(\tilde{V}) - H^*(V)\right|\right|_\infty \leq \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$
- Proof: without loss of generality, let $H^*(\tilde{V})(s) \geq H^*(V)(s)$ and

  let $a_s^* = \underset{a}{\text{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V(s')$

  $\tilde{a}_s^* = \underset{a}{\text{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, \tilde{V}(s')$

Contraction (proof continued)

- Then $0 \leq H^*(\tilde{V})(s) - H^*(V)(s)$ (by assumption)

  $= R(s, \tilde{a}_s^*) + \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*)\, \tilde{V}(s') - R(s, a_s^*) - \gamma \sum_{s'} \Pr(s'|s, a_s^*)\, V(s')$ (by def.)

  $\leq R(s, \tilde{a}_s^*) + \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*)\, \tilde{V}(s') - R(s, \tilde{a}_s^*) - \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*)\, V(s')$

  (since $\tilde{a}_s^*$ suboptimal for V)

  $= \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*)\, [\tilde{V}(s') - V(s')]$

  $\leq \gamma \sum_{s'} \Pr(s'|s, \tilde{a}_s^*) \left|\left|\tilde{V} - V\right|\right|_\infty$ (maxnorm upper bound)

  $= \gamma \left|\left|\tilde{V} - V\right|\right|_\infty$ (since $\sum_{s'} \Pr(s'|s, \tilde{a}_s^*) = 1$)

- Repeat the same argument for $H^*(V)(s) \geq H^*(\tilde{V})(s)$ and for each s

Convergence

- **Theorem 5**: Value iteration converges to V^* for any initial estimate V

$$\lim_{n\to\infty} H^{*(n)}(V) = V^* \quad \forall V$$

- Proof
  - By definition $V^* = H^{*(\infty)}(0)$, but value iteration computes $H^{*(\infty)}(V)$ for some initial V
  - By Lemma 4, $\left|\left|H^{*(n)}(V) - H^{*(n)}(\tilde{V})\right|\right|_\infty \leq \gamma^n \left|\left|V - \tilde{V}\right|\right|_\infty$
  - Hence, when $n \to \infty$, then $\left|\left|H^{*(n)}(V) - H^{*(n)}(0)\right|\right|_\infty \to 0$ and $H^{*(\infty)}(V) = V^* \;\; \forall V$

Value Iteration

- Even when horizon is infinite, perform finitely many iterations
- Stop when $||V_n - V_{n-1}|| \leq \epsilon$

valueIteration(MDP)

$V_0^* \leftarrow \max_a R^a$; $n \leftarrow 0$

Repeat

$n \leftarrow n + 1$

$V_n \leftarrow \max_a R^a + \gamma T^a V_{n-1}$

Until $||V_n - V_{n-1}||_\infty \leq \epsilon$

Return V_n

Induced Policy

- Since $||V_n - V_{n-1}||_\infty \le \epsilon$, by Theorem 5:
  we know that $||V_n - V^*||_\infty \le \frac{\epsilon}{1-\gamma}$

- But, how good is the stationary policy $\pi_n(s)$ extracted based on V_n?

$$\pi_n(s) = \underset{a}{\operatorname{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_n(s')$$

- How far is V^{π_n} from V^*?

Induced Policy

- **Theorem 6**: $\left|\left|V^{\pi_n} - V^*\right|\right|_\infty \leq \frac{2\epsilon}{1-\gamma}$

- Proof

$$\left|\left|V^{\pi_n} - V^*\right|\right|_\infty = \left|\left|V^{\pi_n} - V_n + V_n - V^*\right|\right|_\infty$$

$$\leq \left|\left|V^{\pi_n} - V_n\right|\right|_\infty + \left|\left|V_n - V^*\right|\right|_\infty \quad (||A+B|| \leq ||A|| + ||B||)$$

$$= \left|\left|H^{\pi_n (\infty)}(V_n) - V_n\right|\right|_\infty + \left|\left|V_n - H^{*(\infty)}(V_n)\right|\right|_\infty$$

$$\leq \frac{\epsilon}{1-\gamma} + \frac{\epsilon}{1-\gamma} = \frac{2\epsilon}{1-\gamma} \quad \text{(by Theorems 2 and 5)}$$

Summary

- Value iteration
  - Simple dynamic programming algorithm
  - Complexity: $O(n|A||S|^2)$
    - Here n is the number of iterations

- Can we optimize the policy directly instead of optimizing the value function and then inducing a policy?
  - Yes: by policy iteration

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