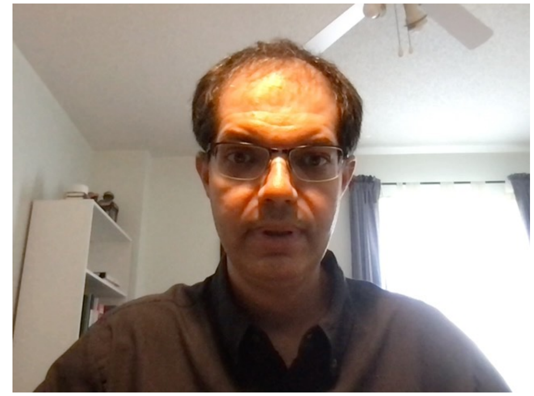


Distributional RL
CS885 Reinforcement Learning
Module 5: October 8, 2021

Bellemare, Marc G., Will Dabney, and Rémi Munos. "A distributional perspective on reinforcement learning." International Conference on Machine Learning. 2017.

Outline

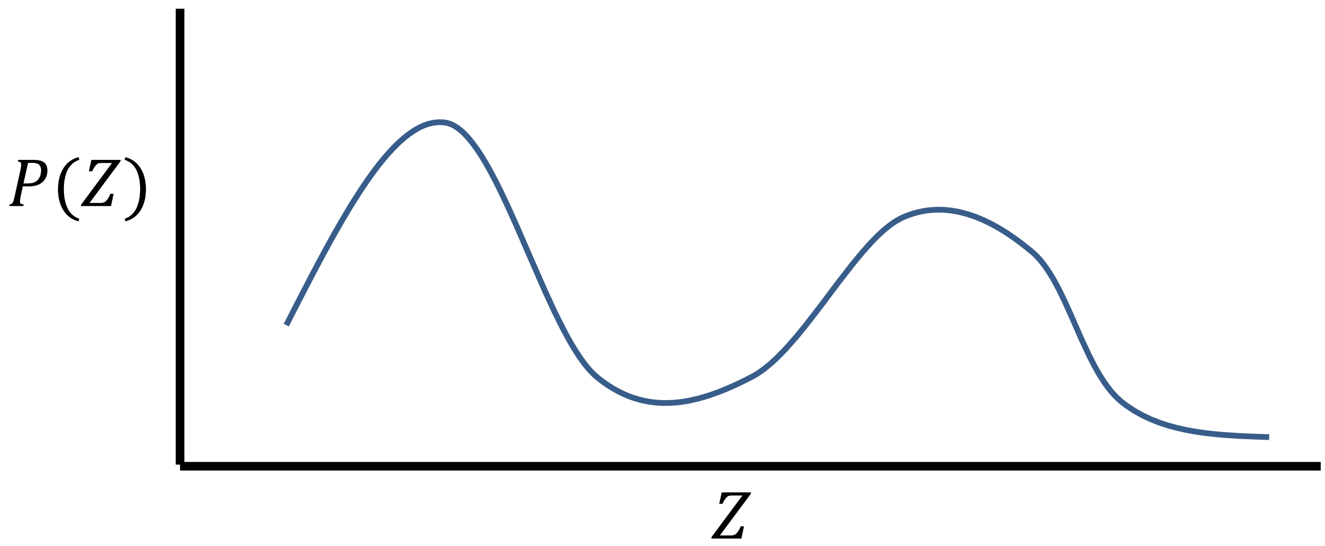
- Distributional Reinforcement Learning
  - Enables risk sensitive objectives
  - Distributional returns
  - C51 (Categorical DQN) Bellemare et al., 2017

Objective

- Let $Z = \sum_t \gamma^t R_t$ be the return random variable
- Traditional RL objective:
  - Mean: $E[Z]$
- Risk sensitive RL objectives:
  - Mean-variance: $E[Z] - \lambda V[Z]$
  - Cumulative distribution: $CDF_Z(z) = \Pr(Z \leq z)$
  - Value at risk: $\mathrm{VaR}_\alpha(Z) = CDF_Z^{-1}(\alpha)$
  - Conditional value at risk: $CVaR_\alpha(Z) = E[Z \mid Z \geq VaR_\alpha(Z)]$

Distributional RL

- Idea: keep track of return distribution $P(Z)$



- Use $P(Z)$ to compute desired objective

Return Distribution

- Random variables:
  $R(s_t, a_t) \sim P(r_t|s_t, a_t)$,
  $s_{t+1} \sim P(s_{t+1}|s_t, a_t)$,
  $a_t \sim \pi(a_t|s_t)$,

- Return distribution:

$$Z^\pi(s_0) = \sum_{t=0} \gamma^t R(s_t, a_t)$$

- Expected return:

$$V^\pi(s_0) = E_{P,\pi}\left[\sum_{t=0} \gamma^t R(s_t, a_t)\right]$$

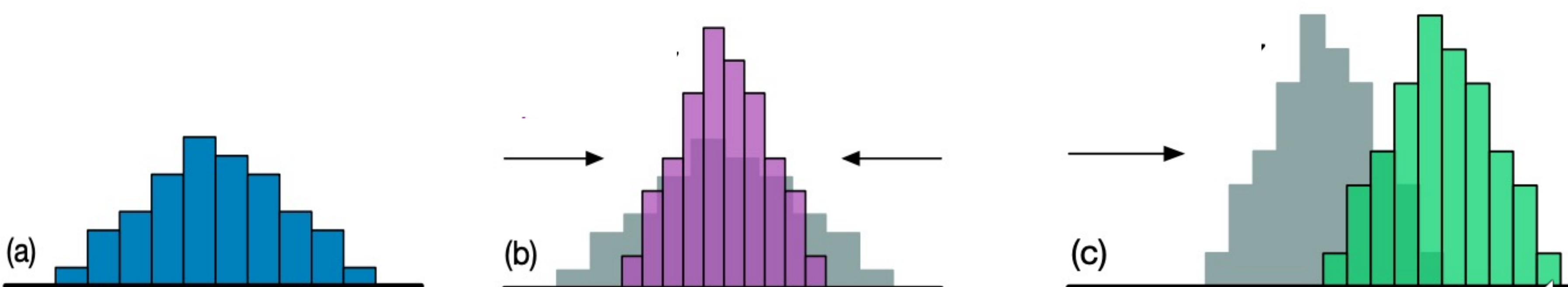
Policy Evaluation

- Policy evaluation

$$Q(s,a) = E_P[R(s,a)] + \gamma E_{P,\pi}[Q(s',a')]$$

$$Z(s,a) = R(s,a) + \gamma Z(s',a')$$

$Z(s',a')$ $\gamma Z(s',a')$ $R(s,a) + \gamma Z(s',a')$



Graphs from Bellemare et al., 2017

Convergence

- Let $\mathcal{T}^\pi$ be the policy evaluation operator
- $\mathcal{T}^\pi Z(s,a) = R(s,a) + \gamma Z(s',a')$

Theorem: $\mathcal{T}^\pi$ converges to a unique return distribution

Proof sketch: $\mathcal{T}^\pi$ is a γ-contraction mapping according to the Wasserstein metric d_w

$$d_W\big(\mathcal{T}^\pi Z(s,a), \mathcal{T}^\pi Z'(s,a)\big) \leq \gamma d_W\big(Z(s,a), Z'(s,a)\big)$$

Bellman Equation

- Bellman Optimality Equation

  $Q(s, a) = E_P[R(s, a)] + \gamma E_{P,\pi}[Q(s', argmax_{a'} Q(s, a'))]$

  $Z(s, a) = R(s, a) + \gamma\, Z(s', argmax_{a'} E[Z(s', a')])$

- NB: cannot replace $argmax_{a'} E[\cdot]$ by risk averse objective since risk averse objectives do not lend themselves to dynamic programming

Convergence

- Let $\mathcal{T}^*$ be the Bellman operator
  $\mathcal{T}^* Z(s,a) = R(s,a) + \gamma Z(s', argmax_{a'} E[Z(s',a')])$

- NB: Optimal return distribution is not unique
  – Each optimal policy may have a different return distribution

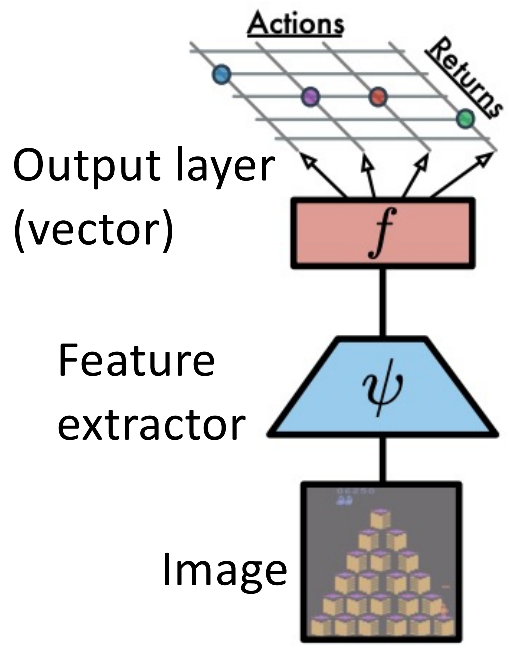
Theorem: $\mathcal{T}^*$ converges to a set of optimal return distributions

Proof: complicated

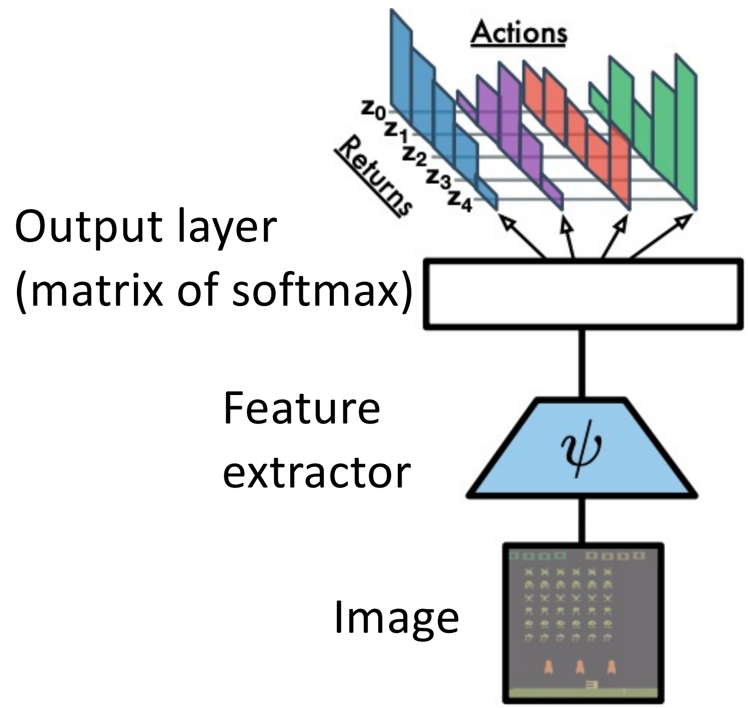
– Cannot show that $\mathcal{T}^*$ is contraction mapping

C51 (Categorical DQN)

DQN



C51 (Categorical DQN)



Pictures from Dabney et al., 2018

C51 (Categorical DQN)

Initialize weights $\boldsymbol{w}$ and $\overline{\boldsymbol{w}}$ at random
Observe current state s
Loop
    Select action a and execute it
    Receive reward r and observe s'
    Add (s, a, s', r) to experience buffer
    Sample mini-batch of experiences from buffer
    For each experience (s, a, s', r) in mini-batch do
        $p_i \leftarrow 0 \quad \forall i \in \{0,1, \ldots, N\}$
        Greedy action: $a' \leftarrow argmax_{a'} \sum_{i'} P_{\overline{\boldsymbol{w}}}(Z(s', a') = z_{i'}) z_{i'}$
        For each $i' \in \{0,1, \ldots, N\}$ do
            Backup $z_{i'}$ and project it in $[z_{min}, z_{max}]$: $\hat{\mathcal{T}} z_{i'} \leftarrow [r + \gamma z_{i'}]_{z_{min}}^{z_{max}}$
            Real index: $i \leftarrow (\hat{\mathcal{T}} z_{i'} - z_{min}) / \Delta z.$ (where $\Delta z = (z_{max} - z_{min})/N$)
            Neighboring integer indices: $l \leftarrow \lfloor i \rfloor, \quad u \leftarrow \lceil i \rceil$
            Distribute probability $P_{\overline{\boldsymbol{w}}}(Z(s', a') = z_{i'})$ of $\hat{\mathcal{T}} z_{i'}$:
                $p_l \leftarrow p_l + P_{\overline{\boldsymbol{w}}}(Z(s', a') = z_{i'})(u - i)$
                $p_u \leftarrow p_u + P_{\overline{\boldsymbol{w}}}(Z(s', a') = z_{i'})(i - l)$
        Cross entropy loss: $L(\boldsymbol{w}) \leftarrow -\sum_i p_i \log P_{\boldsymbol{w}}(Z(s, a) = z_i)$
        Update weights: $\boldsymbol{w} \leftarrow \boldsymbol{w} - \alpha \nabla_{\boldsymbol{w}} L(\boldsymbol{w})$
    Every c steps, update target: $\overline{\boldsymbol{w}} \leftarrow \boldsymbol{w}$

Advantage

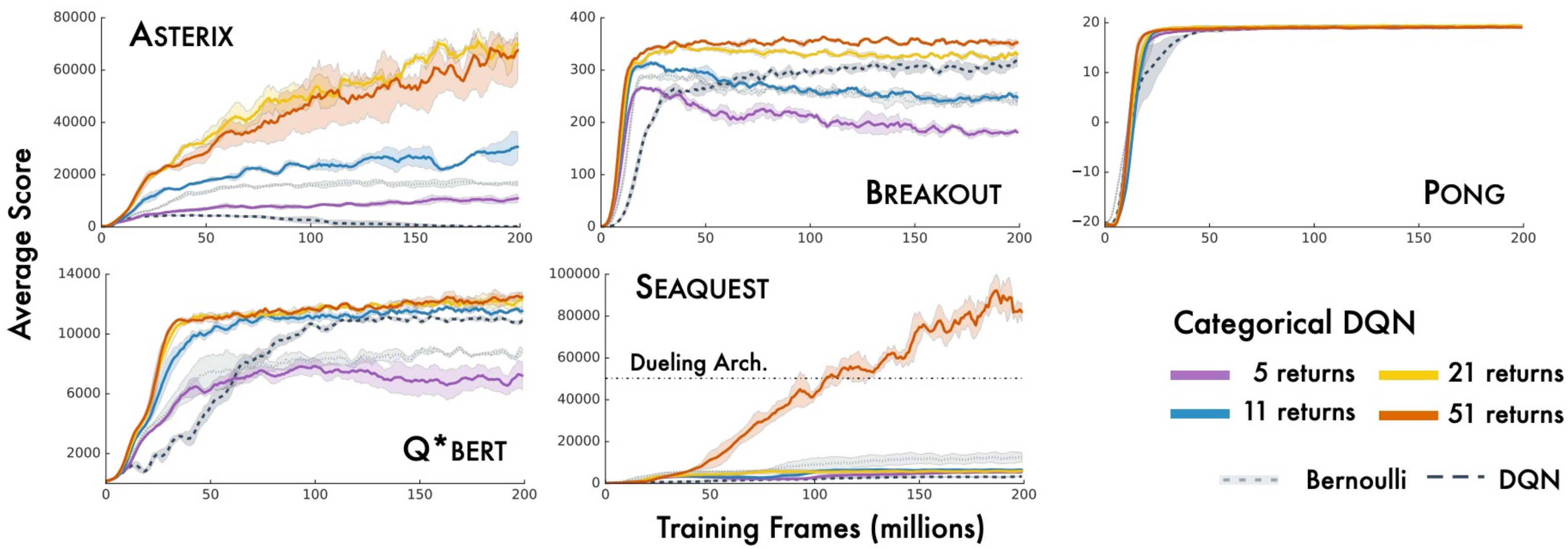
- Why compute a value distribution when the objective is to maximize expected value?

- Categorical DQN can be thought as computing a (weighted) ensemble of returns

$$Q(s,a) = \sum_i P_w(Z(s,a) = z_i) z_i$$

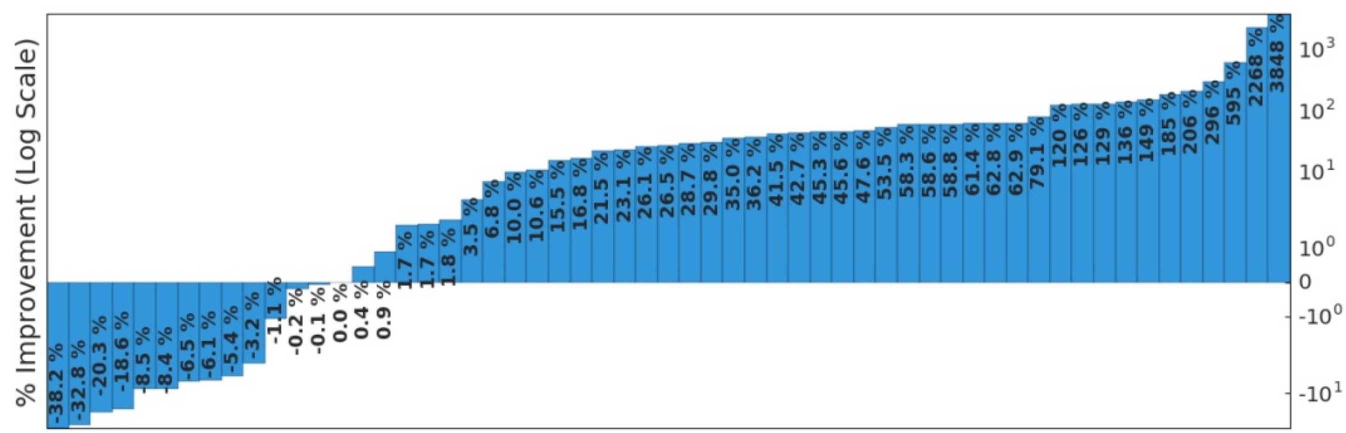
  - Errors in different returns/probabilities may cancel each other, yielding a more accurate estimate of the expectation

Atari Results



Improvement of Categorical DQN over Double DQN

new SOTA in 2017



Graphs from Bellemare et al., 2017

Distributional Representations

- Return distribution:
  - Categorical: C51 (Bellemare et al., 2017), D4PG (Bath-Maron et al., 2018)
  - Samples: VDGL (Freirich et al., 2019) and SDPG (Singh et al., 2020)
- Quantile function (inverse of CDF): $CDF_Z^{-1}(\alpha)$
  - Step function: QR-DQN (Dabney et al., 2018b), IQN (Dabney et al., 2018), FQF (Yang et al., 2019), NC-QR-DQN (Zhou et al., 2020)
  - Piecewise linear: NDQFN (Zhou et al., 2021)
  - Spline: SPL-DQN (Luo et al., 2021)