CS786
Lecture 9: May 30, 2012

Structured Potentials and Weighted Model Counting

[KF] Chapter 5

General Potentials

- General form: table

A	B	$Pr(B\|A)$
a_1	b_1	0.1
a_1	b_2	0.9
a_2	b_1	0.2
a_2	b_2	0.8
a_3	b_1	0.3
a_3	b_2	0.7

Structured Potentials

- Determinism
  - Zeros: **sparse representation**
- Context Specific Independence
  - Repeated values: **tree representation**
- Causal independence
  - Multiplicative decomposition: **factorization**

Determinism

- Degenerate conditional distribution:
  - Deterministic mapping

$$f(A) = \begin{cases} b_1 & \text{if } A = a_3 \\ b_2 & \text{otherwise} \end{cases}$$

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{Pr(B|A)}$
a_1	b_1	0
a_1	b_2	1
a_2	b_1	1
a_2	b_2	0
a_3	b_1	0
a_3	b_2	1

Context Specific Independence

- Recall conditional independence
  $X \perp Y \mid Z \implies \Pr(X|Y,z) = \Pr(X|z)\ \forall z \in dom(Z)$
- What if independence holds only in the context of $Z = z_1$?
  $X \perp Y \mid z_1 \implies \Pr(X|Y,z_1) = \Pr(X|z_1)$
- This is known as **context specific independence**

Repeated Values

- Conditional Independence: $B \perp C \mid A$

A	B	C	$Pr(C\|A,B)$
a_1	b_1	c_1	0.1
a_1	b_1	c_2	0.9
a_1	b_2	c_1	0.1
a_1	b_2	c_2	0.9
a_2	b_1	c_1	0.3
a_2	b_1	c_2	0.7
a_2	b_2	c_1	0.3
a_2	b_2	c_2	0.7

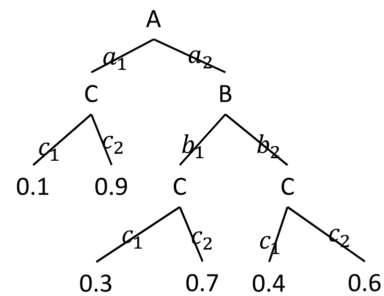
→

A	C	$Pr(C\|A)$
a_1	c_1	0.1
a_1	c_2	0.9
a_2	c_1	0.3
a_2	c_2	0.7

Tree Representation

- Context Specific Independence: $B \perp C \mid a_1$

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{Pr(C \mid A, B)}$
a_1	b_1	c_1	0.1
a_1	b_1	c_2	0.9
a_1	b_2	c_1	0.1
a_1	b_2	c_2	0.9
a_2	b_1	c_1	0.3
a_2	b_1	c_2	0.7
a_2	b_2	c_1	0.4
a_2	b_2	c_2	0.6



Gating

- One parent variable I acts as a gate to determine which cause X_i will be active
$$\Pr(Y \mid I, X_1, X_2, \dots, X_n) = \Pr(Y \mid X_I)$$
- Example:
  - I: sensor activated by a robot
  - $X_1, \dots, X_n$: set of possible sensors
  - Y: noisy sensor reading
- # parameters: linear in # parents
  - Specify $\Pr(Y \mid X_i)\ \ \forall i \in dom(I)$

Chaining

- Cascading sequence of causes

$$\Pr(Y = t | X_1, \dots, X_n) = \begin{cases} 0.1 & if\ X_1 = t \\ 0.2 & if\ X_1 = f \ \wedge\ X_2 = t \\ 0.3 & if\ X_1 = X_2 = f\ \wedge X_3 = t \\ \dots & \\ 0.7 & otherwise \end{cases}$$

- Common in probabilistic if-then-else rules
- # parameters: linear in # parents

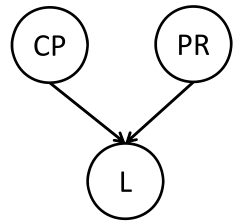
Causal Independence

- Noisy-OR (binary variables)
  – Noisy-MAX (multi-valued variables)
- Noisy-AND (binary variables)
  – Noisy-MIN (multi-valued variables)

- Two common representations
  – Factored representation
  – Augmented Bayesian network

Example: Letter of Reference

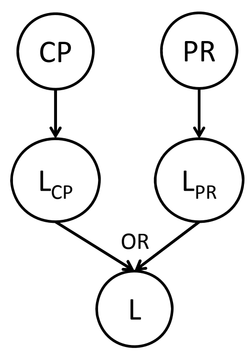
- Variables:
  $ClassParticipation \in \{low, high\}$
  $ProjectReport \in \{good, excellent\}$
  $Letter \in \{average, strong\}$



- Conditional distributions
  $\Pr(L = a|CP = h, PR = g) = 0.1$
  $\Pr(L = a|CP = l, PR = e) = 0.2$
  $\Pr(L = a|CP = h, PR = e) = 0.02$
  $\Pr(L = a|CP = l, PR = g) = 1$

Noisy-OR Formulation

- Re-label domain values
  CP: low=0, high=1
  PR: good=0, excellent=1
  L,L_{CP},L_{PR}: average=0, strong=1

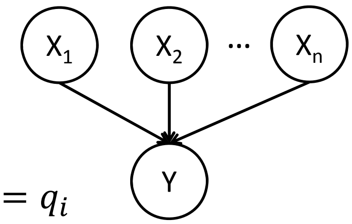


- Conditional distributions for L_{CP},L_{PR}:
  $\Pr(L_{CP} = 0|CP = 0) = 1$
  $\Pr(L_{CP} = 0|CP = 1) = 0.1$
  $\Pr(L_{PR} = 0|PR = 0) = 1$
  $\Pr(L_{PR} = 0|PR = 1) = 0.2$
- Conditional distribution for L:

$$L = L_{CP} \vee L_{PR}$$

Noisy-OR Formulation

- Consider n causes $X_1, \dots, X_n$ that independently affect a variable Y



- Let $\Pr(Y = 0 | X_i = 1, \boldsymbol{X}_{\sim i} = 0) = q_i$
- General formula

$$\Pr(Y = 0 | X) = \prod_i q_i^{X_i}$$

Inference

- How can we exploit local structure in inference?
- Two common approaches
  - Variable elimination with structured potentials (i.e., tree-based representation instead of tables)
  - Weighted model counting

Model Counting

- SAT: Is there a satisfying assignment?
- Model Counting: How many satisfying assignments are there?
- Weighted model counting: What is the sum of the weights of the satisfying assignments?
  - $\Pr(X = x)$: sum of the weights (probabilities) of the joint assignments that satisfy $X = x$?

SAT

- DPLL: backtracking search
  - depth-first search over the space of assignments
  - Backtrack when a clause can never be satisfied
  - Return SAT as soon as all clauses are satisfied
  - Return UNSAT if the search completes without finding any satisfying assignment

Model Counting

- DPLL: backtracking search
  - Depth first search over the space of assignments
- Increment #SAT each time a satisfying assignment is found
- Return #SAT once the search completes

Weighted Model Counting

- DPLL: backtracking search
  - Depth first search over the space of assignments
- Each time a satisfying assignment is found, increment #SAT by the weight of the assignment
- Return #SAT once the search completes

Bayesian Network →Weighted SAT

- Boolean variables:
  - Indicator variables I_{x_i}: for each variable assignment $X = x_i$
  - Parameter variables $\theta_{x_i|pa_X}$: for each parameter
- Weights
  - For each parameter variable:

$$w(\theta_{x_i|pa_X} = true) = \Pr(x_i|pa_X)$$
$$w(\theta_{x_i|pa_X} = false) = 1$$

Clauses

- Indicator clauses:
  - At least one true indicator

$$I_{x_1} \lor I_{x_2} \lor I_{x_3}$$

  - At most one true indicator

$$I_{x_i} \to \neg I_{x_j} \ \forall i \neq j$$

- Parameter clauses:
  - Consistency

$$I_{a_1} \land I_{b_2} \leftrightarrow \theta_{b_2|a_1}$$