CS489/698
Lecture 8: Jan 29, 2018

Perceptrons, Neural Networks

[D] Chapt. 4, [HTF] Chapt. 11, [B] Sec. 4.1.7, 5.1, [M] Sec. 8.5.4, [RN] Sec. 18.7

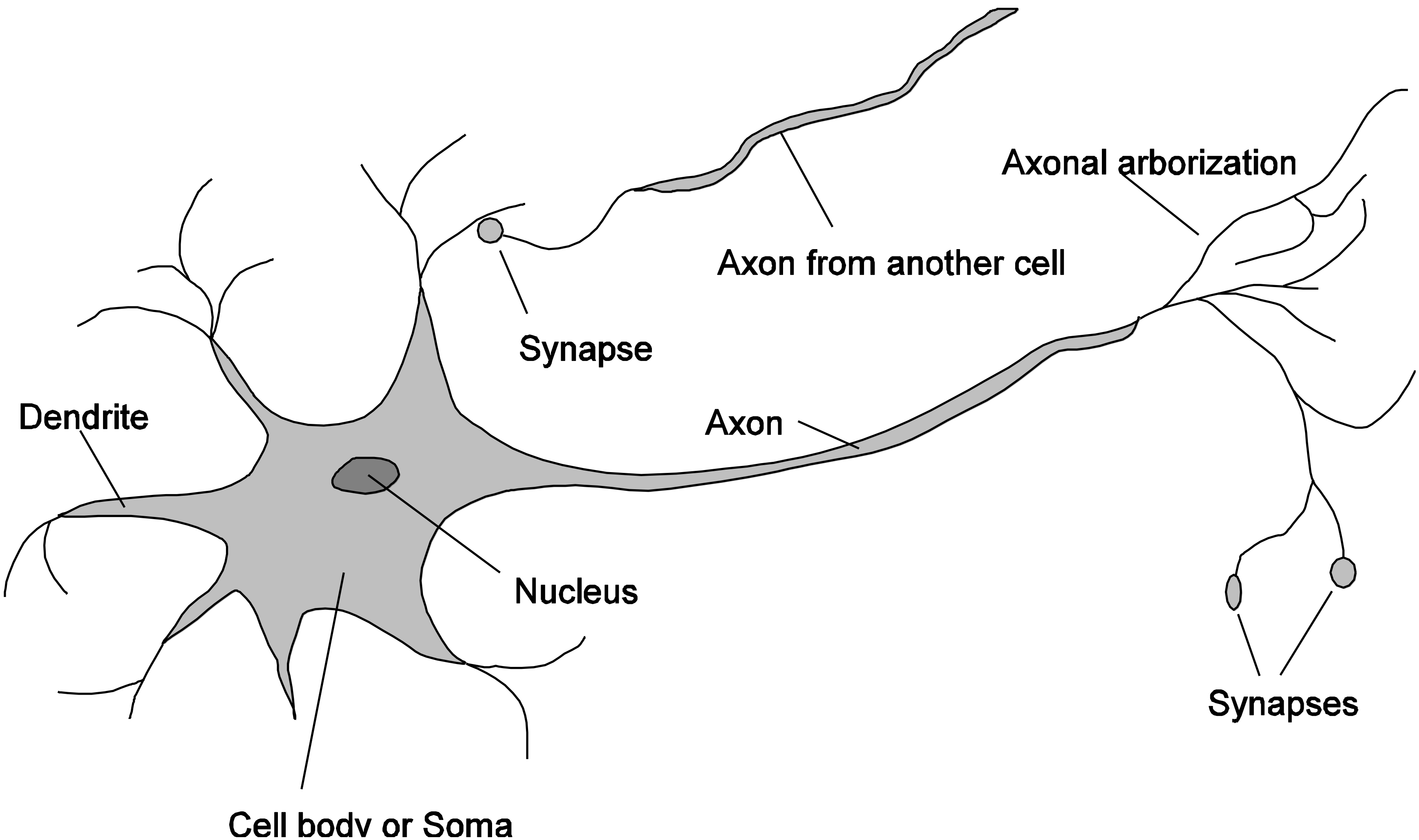
Outline

- Neural networks
  - Perceptron
  - Supervised learning algorithms for neural networks

Brain

- Seat of human intelligence
- Where memory/knowledge resides
- Responsible for thoughts and decisions
- Can learn
- Consists of nerve cells called **neurons**

Neuron


Axonal arborization
Axon from another cell
Synapse
Dendrite
Axon
Nucleus
Synapses
Cell body or Soma

Comparison

- Brain
  - Network of neurons
  - Nerve signals propagate in a neural network
  - Parallel computation
  - **Robust (neurons die everyday without any impact)**
- Computer
  - Bunch of gates
  - Electrical signals directed by gates
  - Sequential and parallel computation
  - **Fragile (if a gate stops working, computer crashes)**

Artificial Neural Networks

- Idea: **mimic the brain to do computation**

- Artificial neural network:
  - Nodes (a.k.a. units) correspond to neurons
  - Links correspond to synapses

- Computation:
  - Numerical signal transmitted between nodes corresponds to chemical signals between neurons
  - Nodes modifying numerical signal corresponds to neurons firing rate

ANN Unit

- For each unit i:
- **Weights: $\boldsymbol{W}$**
  - Strength of the link from unit i to unit j
  - Input signals x_i weighted by W_{ji} and linearly combined: $a_j = \sum_i W_{ji} x_i + w_0 = \boldsymbol{W_j} \; \bar{\boldsymbol{x}}$
- **Activation function: $\boldsymbol{h}$**
  - Numerical signal produced: $y_j = h(a_j)$

ANN Unit

- Picture

Activation Function

- Should be nonlinear
  - Otherwise network is just a linear function
- Often chosen to mimic firing in neurons
  - Unit should be “active” (output near 1) when fed with the “right” inputs
  - Unit should be “inactive” (output near 0) when fed with the “wrong” inputs

Common Activation Functions

Threshold

Sigmoid

Logic Gates

- McCulloch and Pitts (1943)
  - Design ANNs to represent Boolean functions
- What should be the weights of the following units to code AND, OR, NOT ?

AND **OR** **NOT**

Network Structures

- **Feed-forward network**
  - Directed **acyclic** graph
  - No internal state
  - Simply computes outputs from inputs

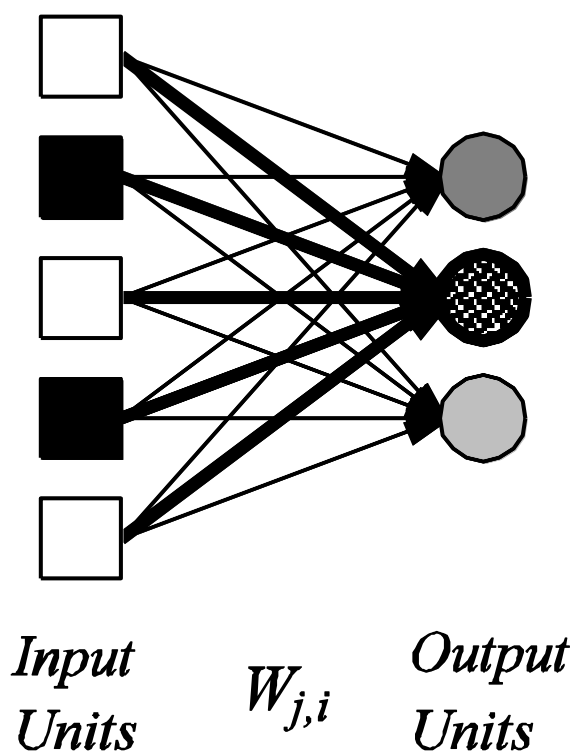
- **Recurrent network**
  - Directed **cyclic** graph
  - Dynamical system with internal states
  - Can memorize information

Feed-forward network

- Simple network with two inputs, one hidden layer of two units, one output unit

Perceptron

- Single layer feed-forward network



Supervised Learning

- Given list of $(\boldsymbol{x}, \boldsymbol{y})$ pairs
- Train feed-forward ANN
  - To compute proper outputs $\boldsymbol{y}$ when fed with inputs $\boldsymbol{x}$
  - Consists of adjusting weights W_{ji}
- **Simple learning algorithm for threshold perceptrons**

Threshold Perceptron Learning

- Learning is done separately for each unit j
  - Since units do not share weights
- Perceptron learning for unit j:
  - For each $(\boldsymbol{x}, y)$ pair do:
    - Case 1: correct output produced
      $$\forall_i \ W_{ji} \leftarrow W_{ji}$$
    - Case 2: output produced is 0 instead of 1
      $$\forall_i \ W_{ji} \leftarrow W_{ji} + x_i$$
    - Case 3: output produced is 1 instead of 0
      $$\forall_i \ W_{ji} \leftarrow W_{ji} - x_i$$
  - Until correct output for all training instances

Threshold Perceptron Learning

- Dot products: $\bar{\boldsymbol{x}}^{\boldsymbol{T}}\bar{\boldsymbol{x}} \geq 0$ and $-\bar{\boldsymbol{x}}^{\boldsymbol{T}}\bar{\boldsymbol{x}} \leq 0$

- Perceptron computes

  1 when $\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}} = \sum_i x_i w_i + w_0 > 0$

  0 when $\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}} = \sum_i x_i w_i + w_0 < 0$
- If output should be 1 instead of 0 then

  $\boldsymbol{w} \leftarrow \boldsymbol{w} + \bar{\boldsymbol{x}}$ since $(\boldsymbol{w} + \bar{\boldsymbol{x}})^T\bar{\boldsymbol{x}} \geq \boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}$
- If output should be 0 instead of 1 then

  $\boldsymbol{w} \leftarrow \boldsymbol{w} - \bar{\boldsymbol{x}}$ since $(\boldsymbol{w} - \bar{\boldsymbol{x}})^T\bar{\boldsymbol{x}} \leq \boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}$

Alternative Approach

- Let $y \in \{-1,1\}\ \forall y$
- Let $M = \{(\boldsymbol{x}_n, y_n)_{\forall n}\}$ be set of misclassified examples
  - i.e., $y_n \boldsymbol{w}^{\boldsymbol{T}} \overline{\boldsymbol{x}}_{\boldsymbol{n}} < 0$

- Find $\boldsymbol{w}$ that minimizes misclassification

$$E(\boldsymbol{w}) = -\sum_{(\boldsymbol{x}_n, y_n) \in M} y_n \boldsymbol{w}^{\boldsymbol{T}} \overline{\boldsymbol{x}}_{\boldsymbol{n}}$$

- Algorithm: gradient descent

$$\boldsymbol{w} \leftarrow \boldsymbol{w} - \eta \boldsymbol{\nabla E}$$

learning rate
or step length

Sequential Gradient Descent

- Gradient: $\boldsymbol{\nabla E} = -\sum_{(x_n, y_n) \in M} y_n \overline{\boldsymbol{x}}_{\boldsymbol{n}}$

- Sequential gradient descent:
  - Adjust $\boldsymbol{w}$ based on one example $(\boldsymbol{x}, y)$ at a time

$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \eta y \overline{\boldsymbol{x}}$$

- When $\eta = 1$, we recover the threshold perceptron learning algorithm

Threshold Perceptron Hypothesis Space

- Hypothesis space $h_{\boldsymbol{w}}$:
  – All binary classifications with parameters $\boldsymbol{w}$ s.t.

$$\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{x}} > 0 \rightarrow +1$$
$$\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{x}} < 0 \rightarrow -1$$

- Since $\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{x}}$ is linear in $\boldsymbol{w}$, perceptron is called a **linear separator**

- **Theorem:** Threshold perceptron learning converges iff the data is linearly separable

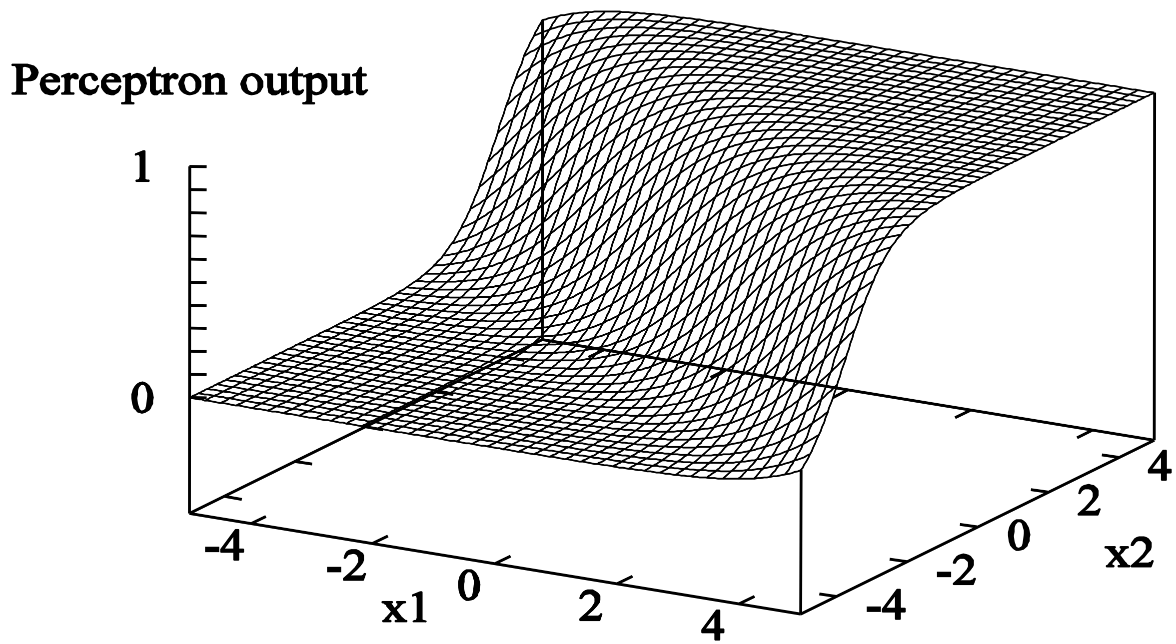
Linear Separability

- Examples:

  Linearly separable   Non-linearly separable

Sigmoid Perceptron

- Represent “soft” linear separators
- **Same hypothesis space as logistic regression**



Sigmoid Perceptron Learning

- Possible objectives
  - **Minimum squared error**

$$E(\boldsymbol{w}) = \frac{1}{2}\sum_n E_n(\boldsymbol{w})^2 = \frac{1}{2}\sum_n \left(y_n - \sigma\left(\boldsymbol{w}^{\boldsymbol{T}}\overline{\boldsymbol{x}}_{\boldsymbol{n}}\right)\right)^2$$

  - **Maximum likelihood**
    - Same algorithm as for logistic regression
  - Maximum a posteriori hypothesis
  - Bayesian Learning

Gradient

- Gradient:

$$\frac{\partial E}{\partial w_i} = \sum_n E_n(w) \frac{\partial E_n}{\partial w_i}$$

$$= -\sum_n E_n(w) \sigma'(w^T \bar{x}_n) x_i$$

Recall that $\sigma' = \sigma(1 - \sigma)$

$$= -\sum_n E_n(w) \sigma(w^T \bar{x}_n)\big(1 - \sigma(w^T \bar{x}_n)\big) x_i$$

Sequential Gradient Descent

- Perceptron-Learning(examples,network)
  - Repeat
    - For each $(\boldsymbol{x_n}, y_n)$ in examples do

$$E_n \leftarrow y_n - \sigma(\boldsymbol{w^T}\boldsymbol{\bar{x}_n})$$

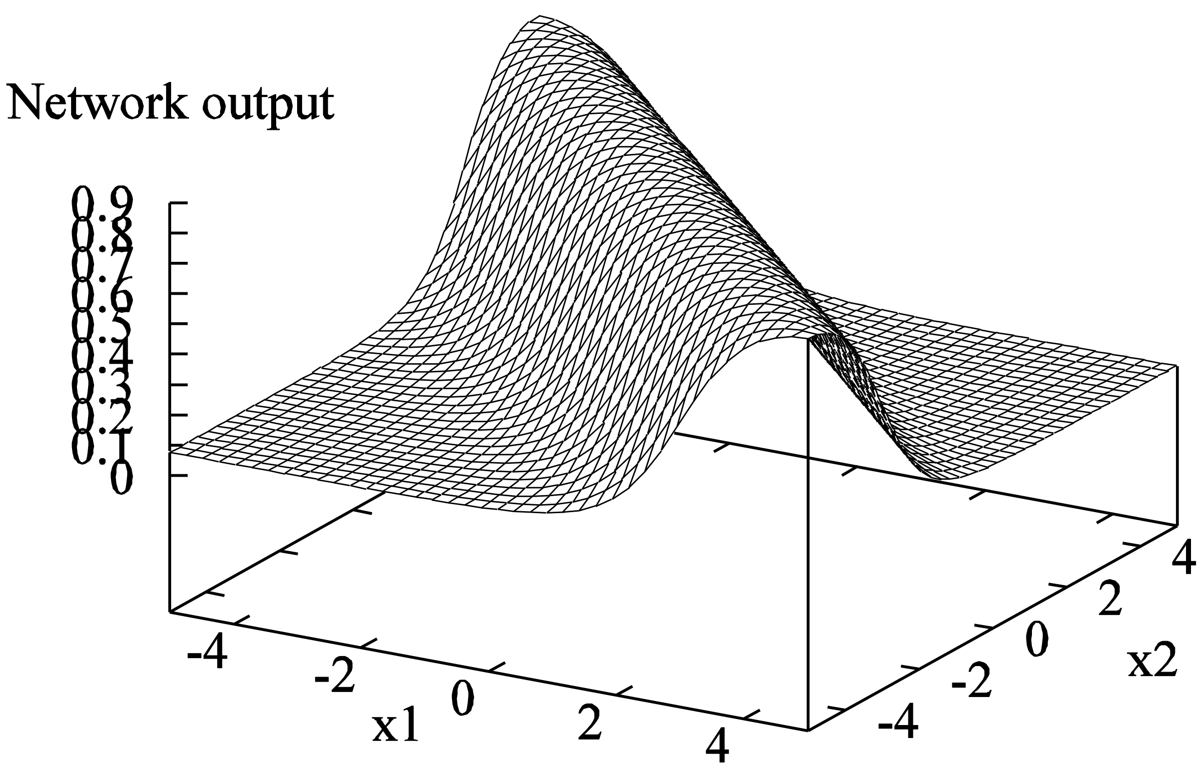
$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \eta\, E_n\, \sigma\left(\boldsymbol{w^T}\boldsymbol{\bar{x}_n}\right)\left(1 - \sigma\left(\boldsymbol{w^T}\boldsymbol{\bar{x}_n}\right)\right)\, \boldsymbol{\bar{x}_n}$$

  - Until some stopping criterion satisfied
  - Return learnt network

- N.B. η is a learning rate corresponding to the step size in gradient descent

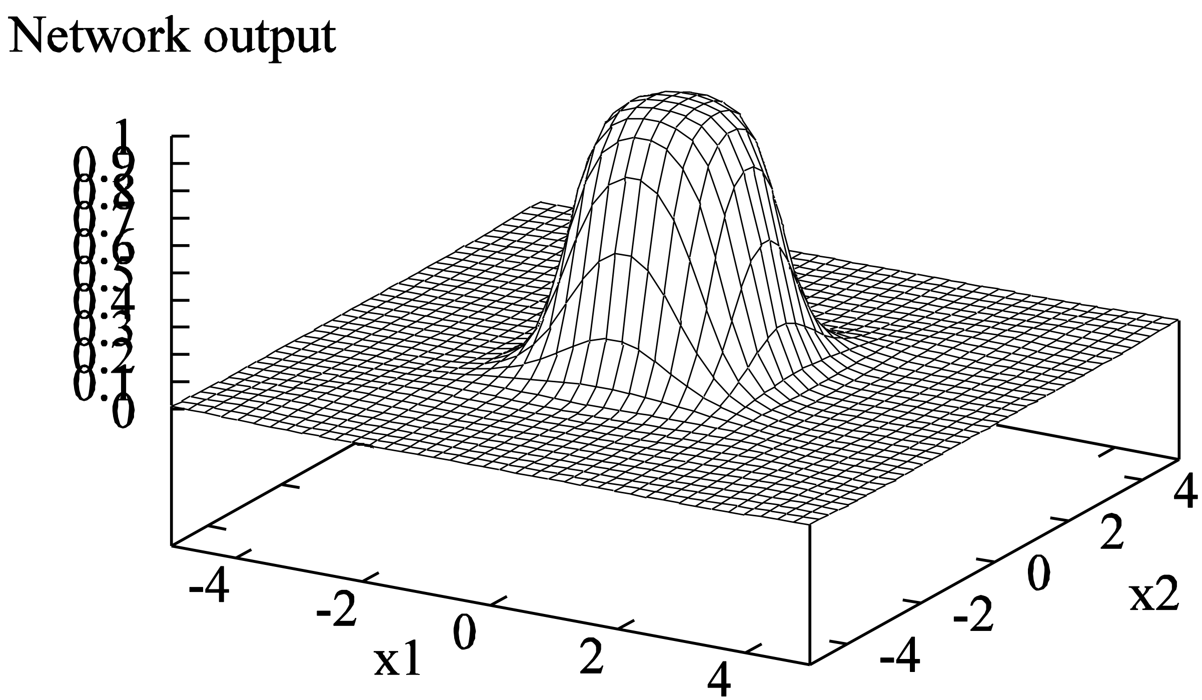
Multilayer Networks

- Adding two sigmoid units with parallel but opposite "cliffs" produces a ridge



Multilayer Networks

- Adding two intersecting ridges (and thresholding) produces a bump



Multilayer Networks

- By tiling bumps of various heights together, we can approximate any function

- Training algorithm:
  - **Back-propagation**
  - Essentially sequential gradient descent performed by propagating errors backward into the network
  - Derivation next class

Neural Net Applications

- Neural nets can approximate any function, hence millions of applications
  - Speech recognition
  - Vision-based object recognition
  - Word embeddings
  - Vision-based autonomous driving
  - Etc.