

# Lecture 4: Constraint Satisfaction

## CS486/686 Intro to Artificial Intelligence

2026-1-15

Pascal Poupart  
David R. Cheriton School of Computer Science

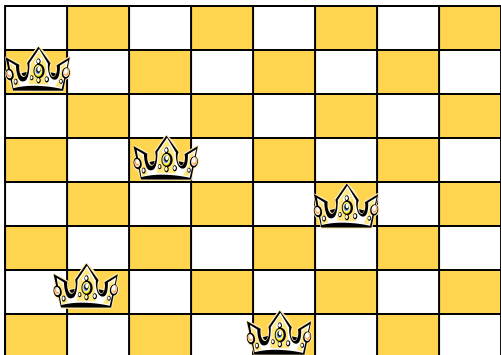
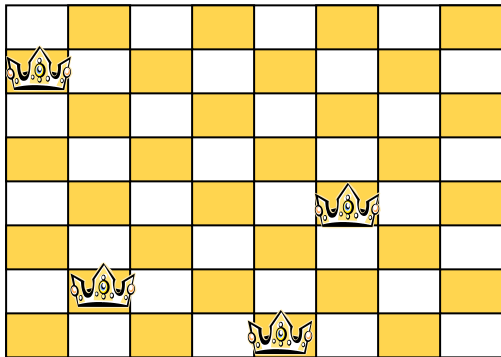
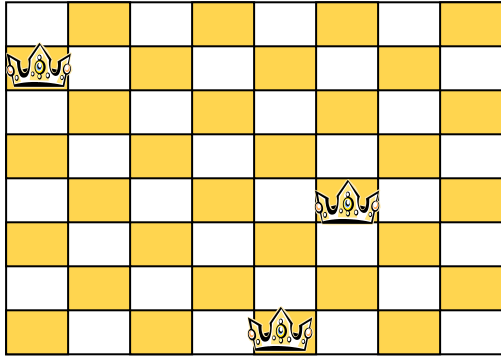


# Outline

- What are Constraint Satisfaction Problems (CSPs)?
- Standard search and CSPs
- Improvements
  - Backtracking
  - Backtracking + heuristics
  - Forward checking

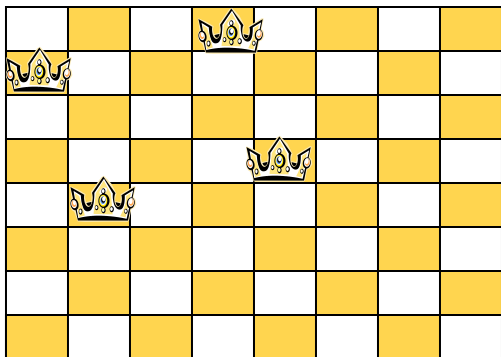
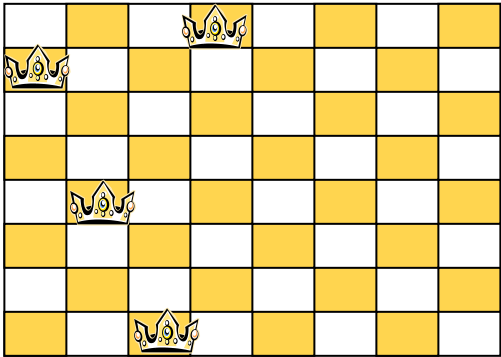
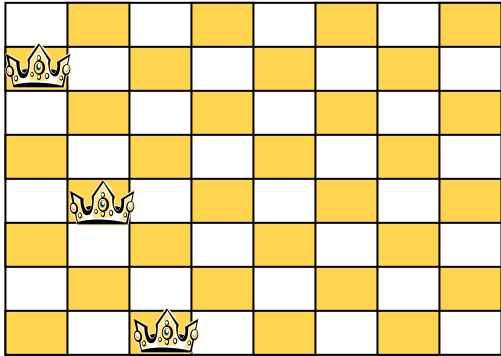
# Introduction

- In the last couple of lectures we have been solving problems by searching in a space of states
  - Treating states as black boxes, ignoring any structure inside them
  - Using problem-specific routines
- Today we study problems where the state structure is important



- **States:** all arrangements of 0,1,..., or 8 queens on the board
- **Initial state:** 0 queens on the board
- **Successor function:** Add a queen to the board
- **Goal test:** 8 queens on the board with no two of them attacking each other

$$64 \times 63 \times \dots \times 57 \approx 3 \times 10^{14} \text{ states}$$



- **States:** all arrangements  $k$  queens ( $0 \leq k \leq 8$ ), one per column in the leftmost  $k$  columns, with no queen attacking another
- **Initial state:** 0 queens on the board
- **Successor function:** Add a queen to the leftmost empty column such that it is not attacked
- **Goal test:** 8 queens on the board

2057 States

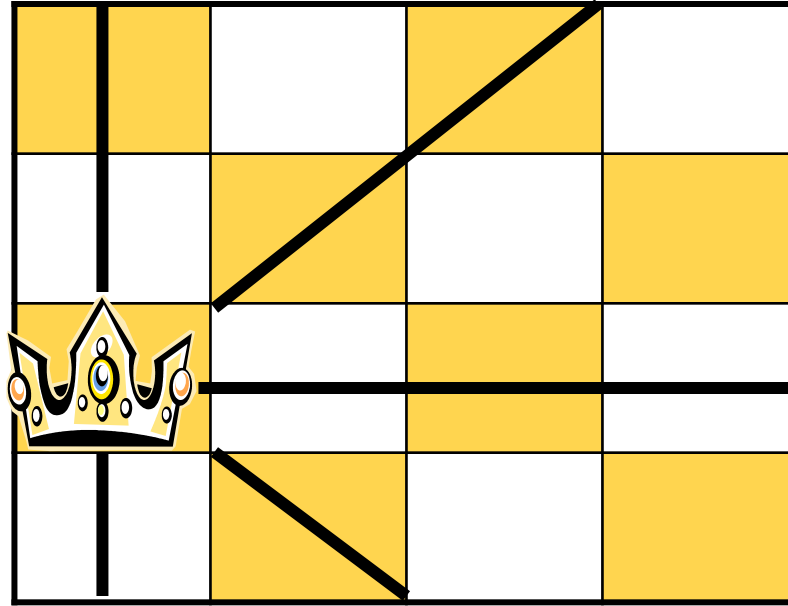
# Introduction

- Earlier search methods studied often make choices in an arbitrary order
- In many problems the same state can be reached independent of the order in which the moves are chosen (commutative actions)
- Can we solve problems efficiently by being smart in the order in which we take actions?

# 4-Queens Constraint Propagation

Place a queen in a square

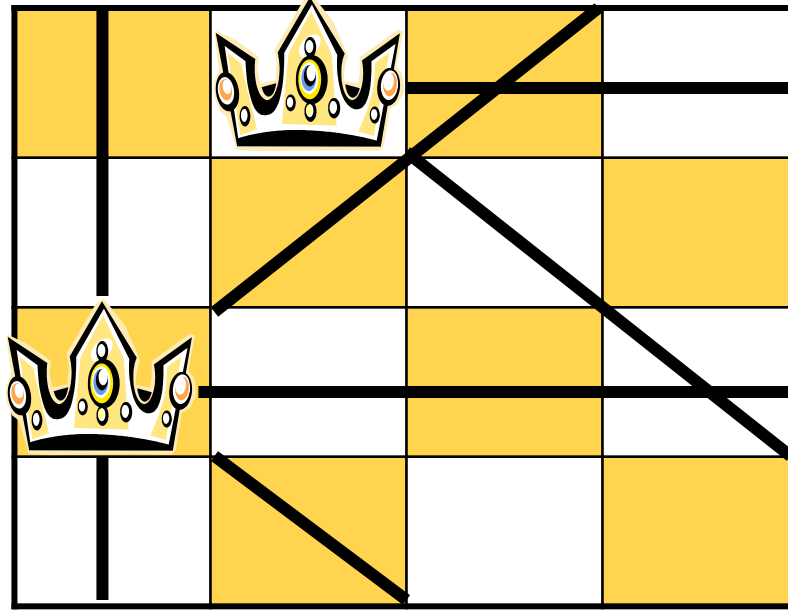
Remove conflicting squares  
from consideration



# 4-Queens Constraint Propagation

Place a queen in a square

Remove conflicting squares  
from consideration

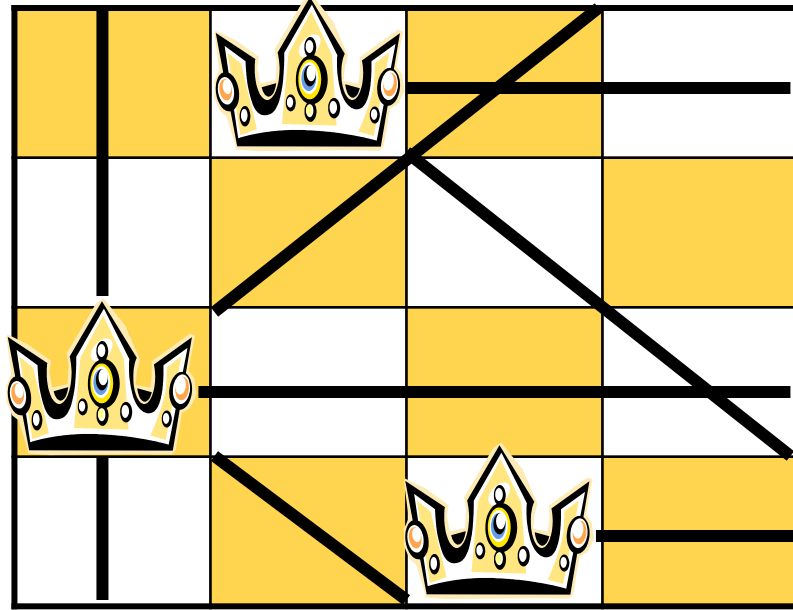




# 4-Queens Constraint Propagation

Place a queen in a square

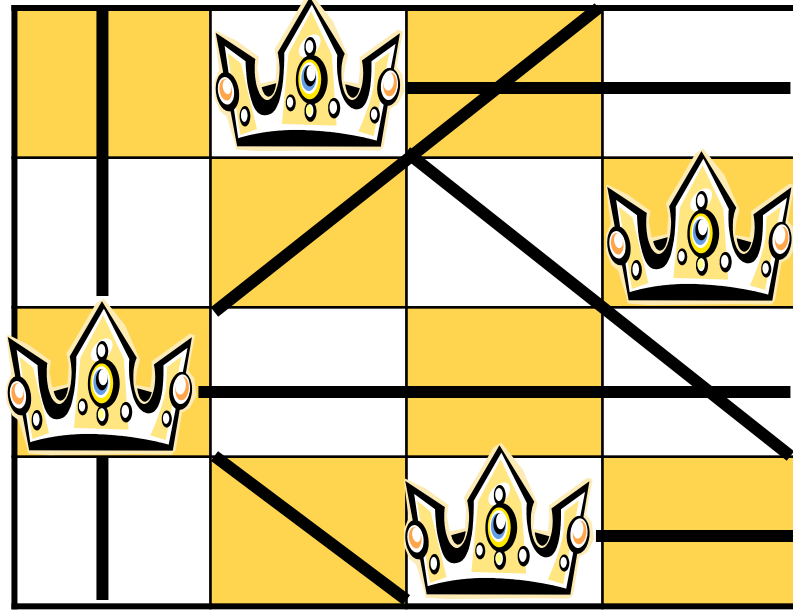
Remove conflicting squares  
from consideration



# 4-Queens Constraint Propagation

Place a queen in a square

Remove conflicting squares  
from consideration



# CSP Definition

- A **constraint satisfaction problem** (CSP) is defined by  $\{V, D, C\}$  where
  - $V = \{V_1, V_2, \dots, V_n\}$  is a set of variables
  - $D = \{D_1, \dots, D_n\}$  is the set of **domains**,  $D_i$  is the domain of possible values for variable  $V_i$
  - $C = \{C_1, \dots, C_m\}$  is the set of **constraints**
    - Each constraint involves some subset of the variables and specifies the allowable combinations of values for that subset

# CSP Definition

- A state is an **assignment** of values to some or all of the variables

$$\{V_i = x_i, V_j = x_j, \dots\}$$

- An assignment is **consistent** if it does not violate any constraints
- A **solution** is a complete, consistent assignment (“hard constraints”)
  - Some CSPs also require an objective function to be optimized (“soft constraints”)

# Example 1: 8-Queens

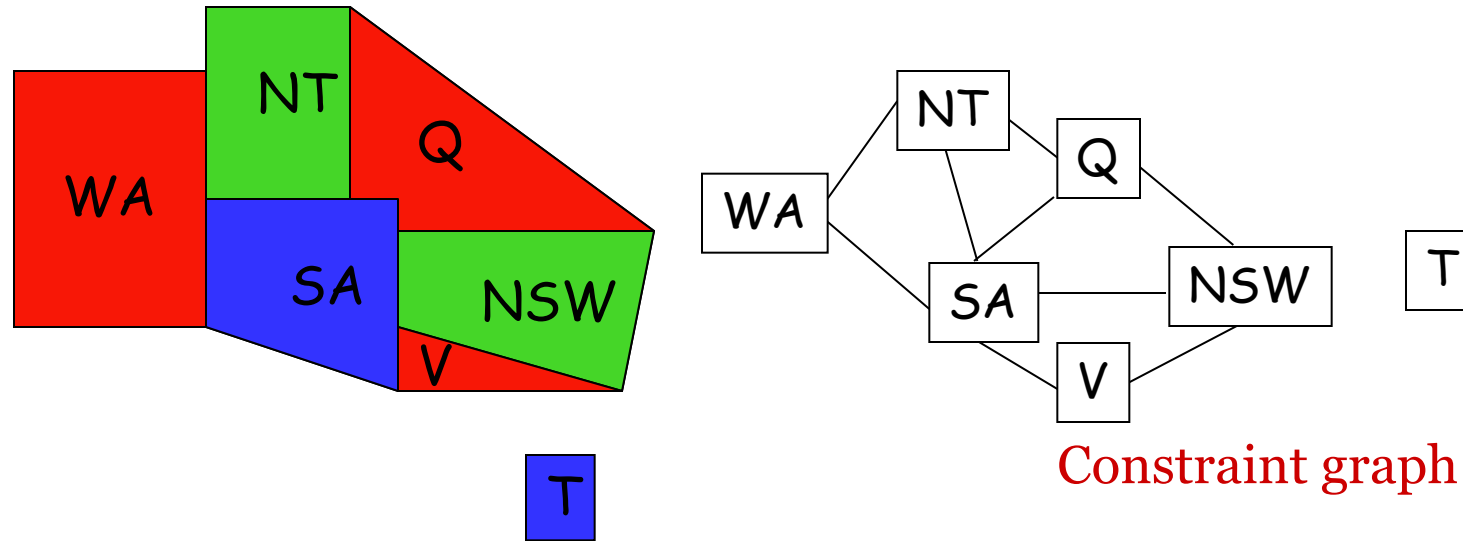
- 64 variables  $V_{ij}$ ,  $i = 1$  to  $8$ ,  $j = 1$  to  $8$
- Domain of each variable is  $\{0,1\}$
- Constraints
  - $V_{ij} = 1 \rightarrow V_{ik} = 0$  for all  $k \neq j$
  - $V_{ij} = 1 \rightarrow V_{kj} = 0$  for all  $k \neq i$
  - Similar constraint for diagonals
  - $\sum_{ij} V_{ij} = 8$

Binary constraints  
relate two variables

## Example 2 – 8 queens

- 8 variables  $V_i$ ,  $i = 1$  to 8
- Domain of each variable is  $\{1, 2, \dots, 8\}$
- Constraints
  - $V_i = k \rightarrow V_j \neq k$  for all  $j \neq i$
  - Similar constraints for diagonals

# Example 3 - Map Coloring



- 7 variables  $\{WA, NT, SA, Q, NSW, V, T\}$
- Each variable has the same domain:  $\{red, green, blue\}$
- No two adjacent variables have the same value:

$$WA \neq NT, WA \neq SA, NT \neq SA, NT \neq Q, SA \neq Q, SA \neq NSW, SA \neq V, Q \neq NSW, NSW \neq V$$

# Example 4 - Street Puzzle

The Englishman lives in the Red house  
The Spaniard has a Dog  
The Japanese is a Painter  
The Italian drinks Tea  
The Norwegian lives in the first house on the left  
The owner of the Green house drinks Coffee  
The Green house is on the right of the White house  
The Sculptor breeds Snails  
The Diplomat lives in the Yellow house  
The owner of the middle house drinks Milk  
The Norwegian lives next door to the Blue house  
The Violinist drinks Fruit juice  
The Fox is in the house next to the Doctor's  
The Horse is next to the Diplomat's

1 2 3 4 5

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

Who owns the Zebra?  
Who drinks Water?



# Example 4 - Street Puzzle

1 2 3 4 5

$$(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$$

The Englishman lives in the Red house

The Spaniard has a Dog

The Japanese is a Painter  $\rightarrow (N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$

The Italian drinks Tea

The Norwegian lives in the first house on the left  $\rightarrow (N_1 = \text{Norwegian})$

The owner of the Green house drinks Coffee

The Green house is on the right of the White house  $\rightarrow \begin{cases} (C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green}) \\ (C_5 \neq \text{White}) \\ (C_1 \neq \text{Green}) \end{cases}$

The Sculptor breeds Snails

The Diplomat lives in the Yellow house

The owner of the middle house drinks Milk

The Norwegian lives next door to the Blue house  $\rightarrow$  left as an exercise

The Violinist drinks Fruit juice

The Fox is in the house next to the Doctor's

The Horse is next to the Diplomat's

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

# Example 4 - Street Puzzle

1 2 3 4 5

$$(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$$

The Englishman lives in the Red house

The Spaniard has a Dog

The Japanese is a Painter  $\rightarrow (N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$

The Italian drinks Tea

The Norwegian lives in the first house on the left  $\rightarrow (N_1 = \text{Norwegian})$

The owner of the Green house drinks Coffee

The Green house is on the right of the White house  $\rightarrow \begin{cases} (C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green}) \\ (C_5 \neq \text{White}) \\ (C_1 \neq \text{Green}) \end{cases}$

The Sculptor breeds Snails

The Diplomat lives in the Yellow house

The owner of the middle house drinks Milk

The Norwegian lives next door to the Blue house

The Violinist drinks Fruit juice

The Fox is in the house next to the Doctor's

The Horse is next to the Diplomat's

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

unary constraints

# Example 4 - Street Puzzle

The Englishman lives in the Red house  
The Spaniard has a Dog  
The Japanese is a Painter  
The Italian drinks Tea  
The Norwegian lives in the first house on the left  
The owner of the Green house drinks Coffee  
The Green house is on the right of the White house  
The Sculptor breeds Snails  
The Diplomat lives in the Yellow house  
The owner of the middle house drinks Milk  
The Norwegian lives next door to the Blue house  
The Violinist drinks Fruit juice  
The Fox is in the house next to the Doctor's  
The Horse is next to the Diplomat's

1 2 3 4 5

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

Implicit constraints:

$\forall i, j \in [1, 5], i \neq j, N_i \neq N_j$

$\forall i, j \in [1, 5], i \neq j, C_i \neq C_j$

...

# Example 4 - Street Puzzle

1 2 3 4 5

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

The Englishman lives in the Red house

The Spaniard has a Dog

The Japanese is a Painter

The Italian drinks Tea

The Norwegian lives in the first house on the left  $\rightarrow N_1 = \text{Norwegian}$

The owner of the Green house drinks Coffee

The Green house is on the right of the White house

The Sculptor breeds Snails

The Diplomat lives in the Yellow house

The owner of the middle house drinks Milk  $\rightarrow D_3 = \text{Milk}$

The Norwegian lives next door to the Blue house

The Violinist drinks Fruit juice

The Fox is in the house next to the Doctor's

The Horse is next to the Diplomat's

Use inference to  
derive new facts

# Example 4 - Street Puzzle

1 2 3 4 5

$N_i = \{\text{English, Spaniard, Japanese, Italian, Norwegian}\}$

$C_i = \{\text{Red, Green, White, Yellow, Blue}\}$

$D_i = \{\text{Tea, Coffee, Milk, Fruit-juice, Water}\}$

$J_i = \{\text{Painter, Sculptor, Diplomat, Violinist, Doctor}\}$

$A_i = \{\text{Dog, Snails, Fox, Horse, Zebra}\}$

The Englishman lives in the Red house  $\rightarrow C_1 \neq \text{Red}$

The Spaniard has a Dog  $\rightarrow A_1 \neq \text{Dog}$

The Japanese is a Painter

The Italian drinks Tea

The Norwegian lives in the first house on the left  $\rightarrow N_1 = \text{Norwegian}$

The owner of the Green house drinks Coffee

The Green house is on the right of the White house

The Sculptor breeds Snails

The Diplomat lives in the Yellow house

The owner of the middle house drinks Milk  $\rightarrow D_3 = \text{Milk}$

The Norwegian lives next door to the Blue house

The Violinist drinks Fruit juice  $\rightarrow J_3 \neq \text{Violinist}$

The Fox is in the house next to the Doctor's

The Horse is next to the Diplomat's

Use inference to  
derive new facts

# Example 5 - Scheduling

Four tasks  $T_1$ ,  $T_2$ ,  $T_3$ , and  $T_4$  are related by time constraints:

- $T_1$  must be done during  $T_3$
  - $T_2$  must be achieved before  $T_1$  starts
  - $T_2$  must overlap with  $T_3$
  - $T_4$  must start after  $T_1$  is complete
- Are the constraints compatible?
  - What are the possible time relations between two tasks?
  - What if the tasks use resources in limited supply?

# Example 6 - 3-Sat

- $n$  Boolean variables,  $V_1, \dots, V_n$
- $K$  constraints of the form  $V_i \vee V_j \vee V_k$  where  $V_i$  is either true or false
- NP-complete

# Properties of CSPs

- Types of variables
  - Discrete and finite
    - Map colouring, 8-queens, Boolean CSPs
  - Discrete variables with infinite domains
    - Scheduling jobs in a calendar
    - Require a **constraint language** ( $Job_1 + 3 \leq Job_2$ )
  - Continuous domains
    - Scheduling on the Hubble telescope
    - **Linear programming**



# Properties of CSPs

- Types of constraints
  - **Unary** constraint relates a single variable to a value
    - *Queensland = Blue, SA  $\neq$  Green*
  - **Binary** constraint relates two variables
    - *SA  $\neq$  NSW*
    - Can use a constraint graph to represent CSPs with only binary constraints
  - Higher order constraints involve three or more variables
    - *Alldiff( $V_1, \dots, V_n$ )*
    - Can use a constraint hypergraph to represent the problem

# CSPs and search

- $N$  variables  $V_1, \dots, V_n$
- Valid assignment:  $\{V_1 = x_1, \dots, V_k = x_k\}$  for  $0 \leq k \leq n$  such that values satisfy constraints on the variables
- States: valid assignments
- Initial state: empty assignment
- Successor:  $\{V_1 = x_1, \dots, V_k = x_k\} \rightarrow \{V_1 = x_1, \dots, V_k = x_k, V_{k+1} = x_{k+1}\}$
- Goal test: complete assignment
- If all domains have size  $d$ , then there are  $O(d^n)$  complete assignments

# CSPs and commutativity

- CSPs are commutative!
  - The order of application of any given set of actions has no effect on the outcome
  - When assigning values to variables we reach the same partial assignment, no matter the order
- All CSP search algorithms generate successors by considering possible assignments for only a **single variable at each node in the search tree**

# CSPs and commutativity

- 3 variables  $V_1, V_2, V_3$
- Let the current assignment be  $A = \{V_1 = x_1\}$
- Pick variable  $V_3$
- Let domain of  $V_3$  be  $\{a, b, c\}$
- The successors of  $A$  are

$$\{V_1 = x_1, V_3 = a\}$$

$$\{V_1 = x_1, V_3 = b\}$$

$$\{V_1 = x_1, V_3 = c\}$$

# Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return RECURSIVE-BACKTRACKING({}, csp)

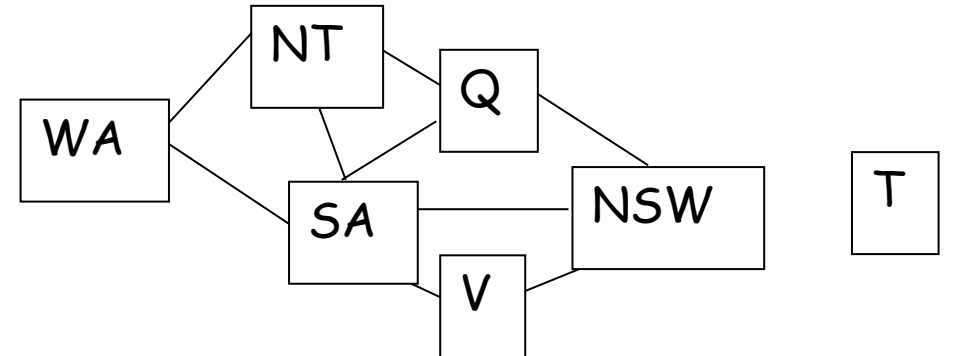
function RECURSIVE-BACKTRACKING(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(Variables[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment according to Constraints[csp] then
      add { var = value } to assignment
      result ← RECURSIVE-BACKTRACKING(assignment, csp)
      if result ≠ failure then return result
      remove { var = value } from assignment
  return failure
```

Depth first search that chooses values for one variable at a time

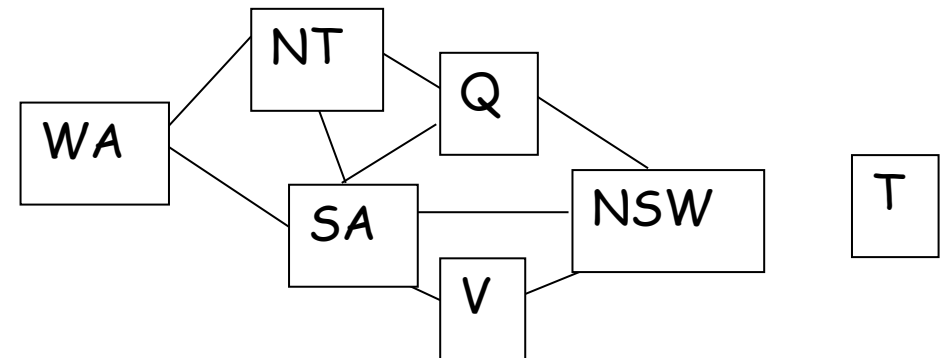
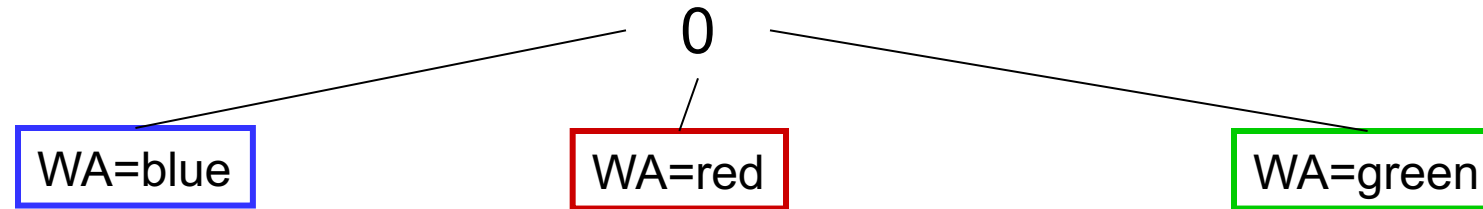
Backtracks when a variable has no legal values to assign

# Backtracking

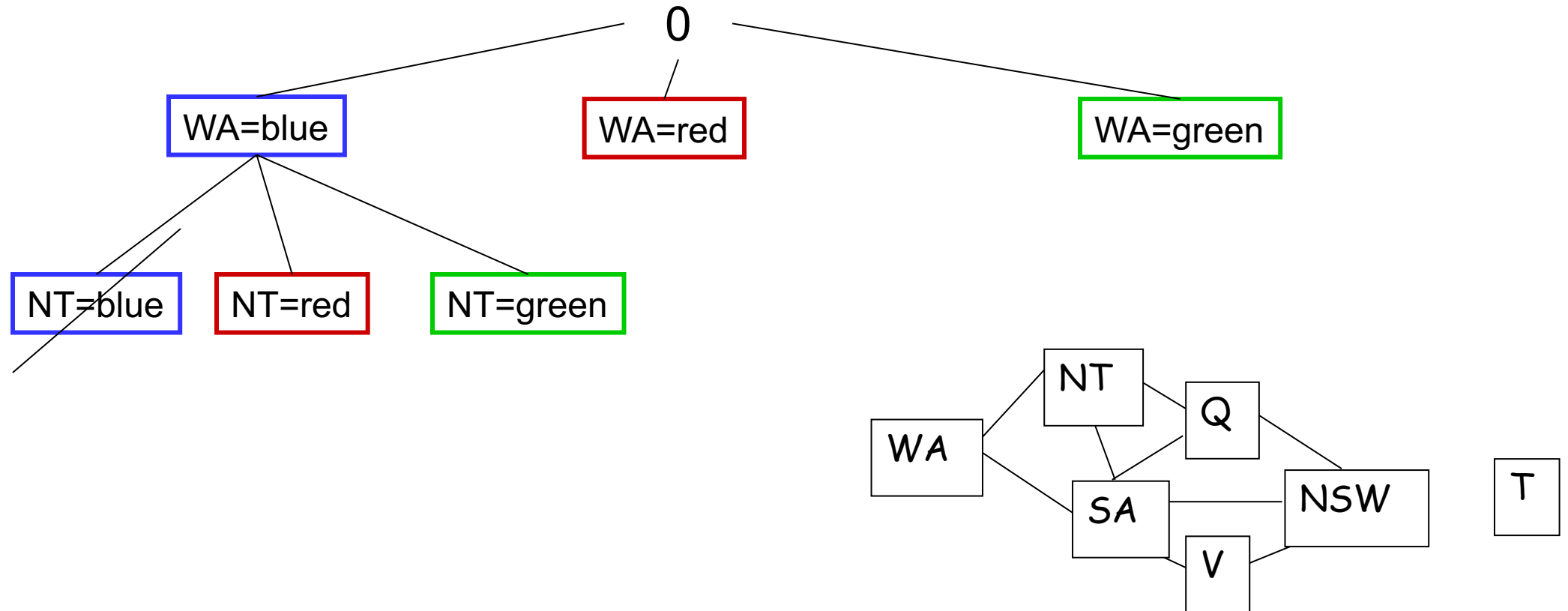
0



# Backtracking

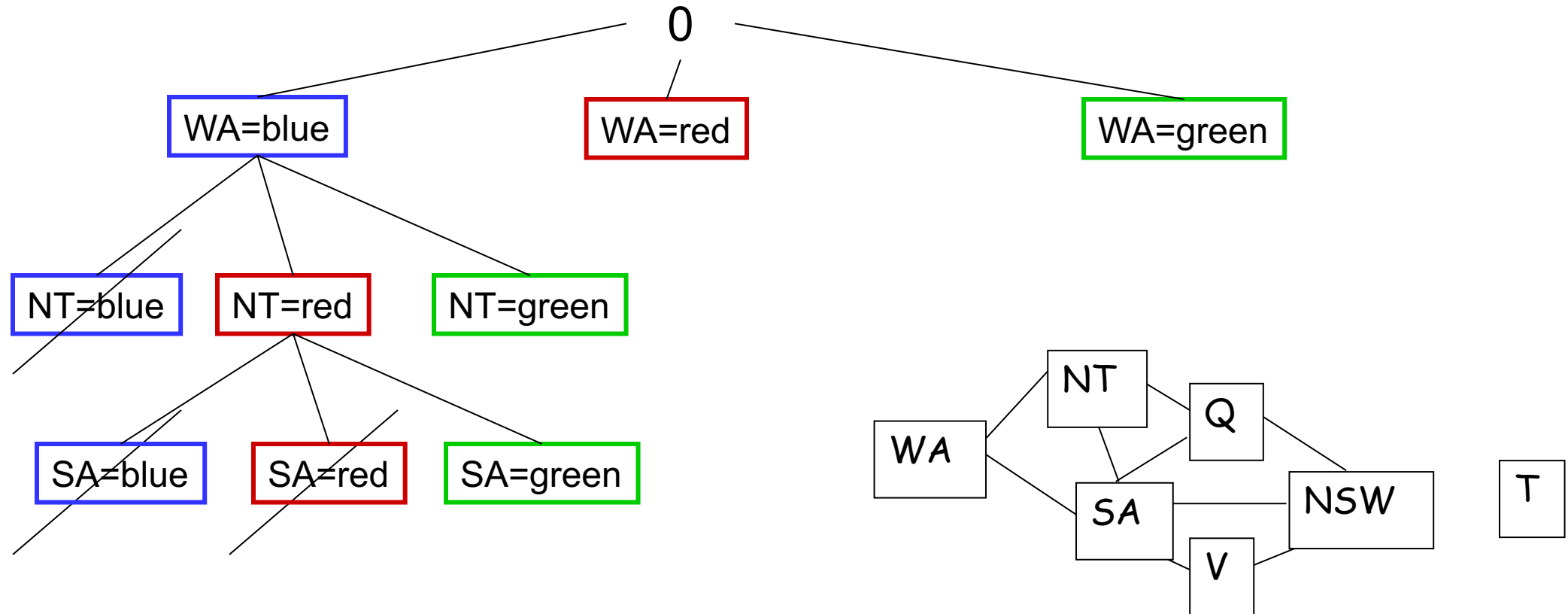


# Backtracking





# Backtracking



# Backtracking and efficiency

- Backtracking search is an uninformed search method
  - Not very efficient
- We can do better by thinking about the following questions
  - Which variable should be assigned next?
  - In which order should its values be tried?
  - Can we detect inevitable failure early (and avoid same failure in other paths)?

# Most constrained variable

- Choose the variable which has the fewest “legal” moves
  - AKA minimum remaining values (MRV) heuristic



$$D_{NT} = \{green, blue\}$$

$$D_{SA} = \{blue\}$$

$$D_{SA} = \{green, blue\}$$

$$D_Q = \{blue, red\}$$

$$D_{others} = \{red, green, blue\}$$

$$D_{others} = \{red, green, blue\}$$

# Most constraining variable

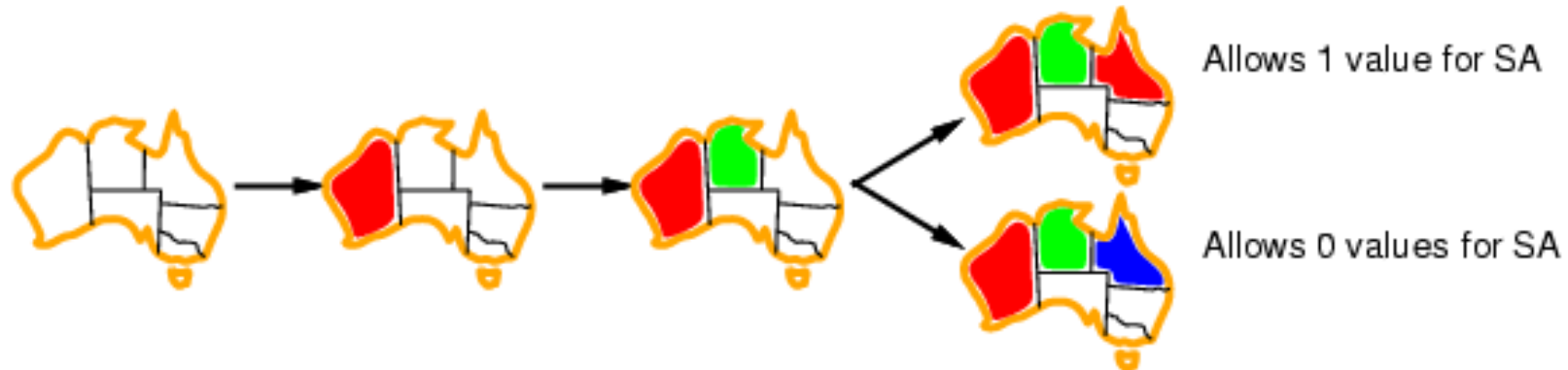
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables
- Tie-breaker among most constrained variables



SA is involved in 5 constraints

# Least-constraining value

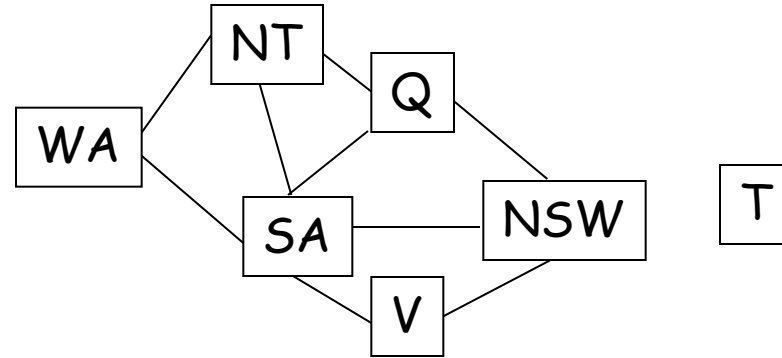
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



# Forward checking

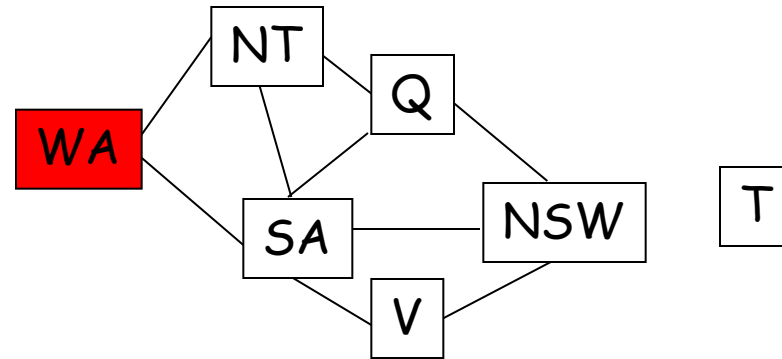
- The third question was
  - Is there a way to detect failure early?
- Forward checking
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

# Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

# Forward Checking in Map Coloring

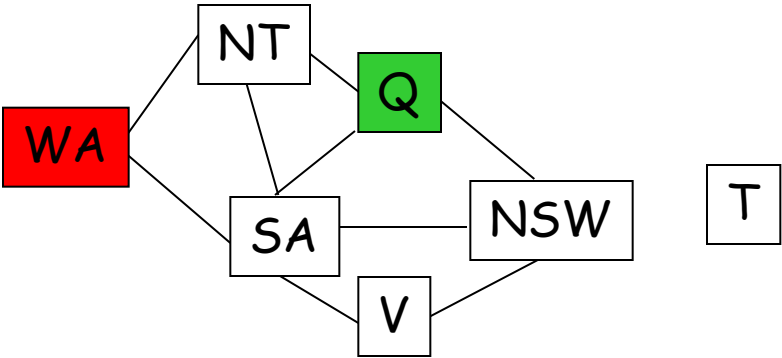


WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	<del>RGB</del>	RGB	RGB	RGB	<del>RGB</del>	RGB

Forward checking removes the value Red of NT and of SA

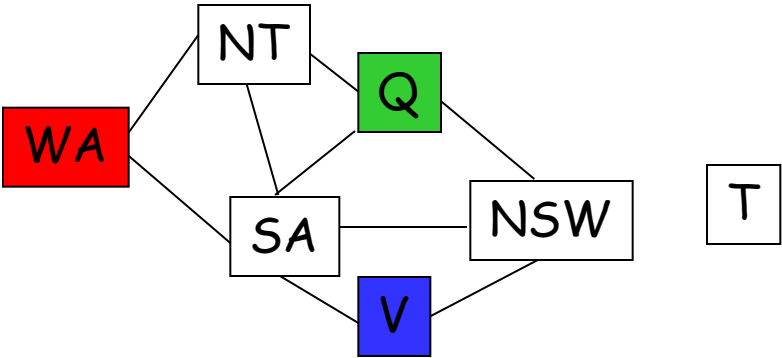


# Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	<del>GB</del>	G	<del>RGB</del>	RGB	<del>GB</del>	RGB

# Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	<del>RB</del>	B	<del>B</del>	RGB

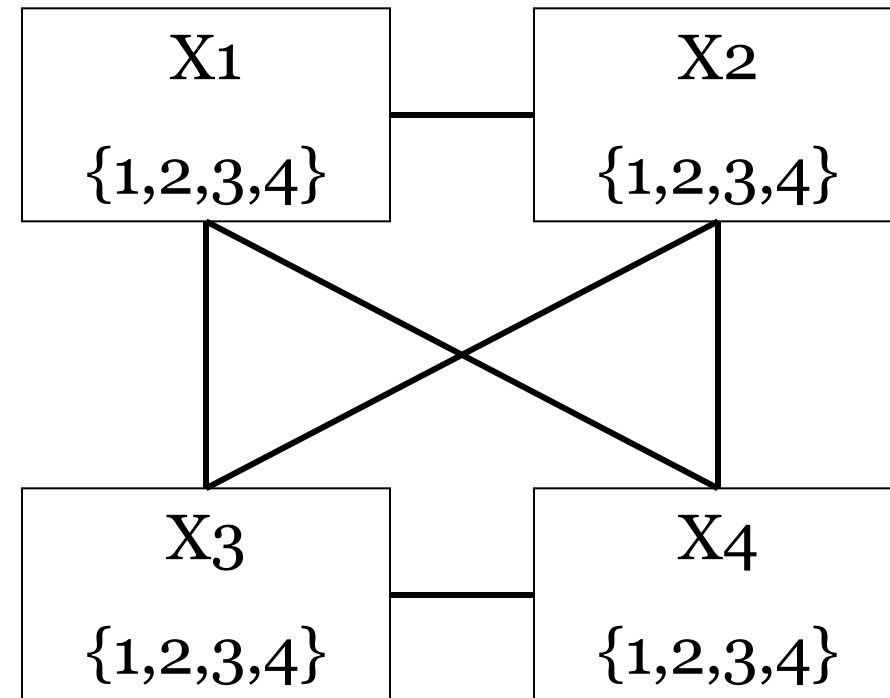
# Forward Checking in Map Coloring

Empty set: the current assignment  
 $\{(WA \leftarrow R), (Q \leftarrow G), (V \leftarrow B)\}$   
does not lead to a solution

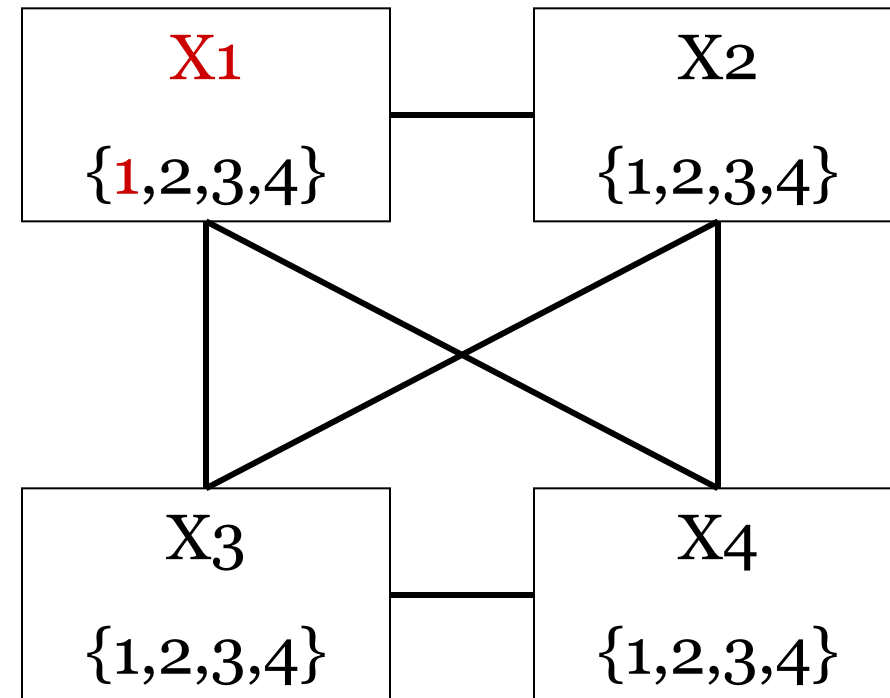
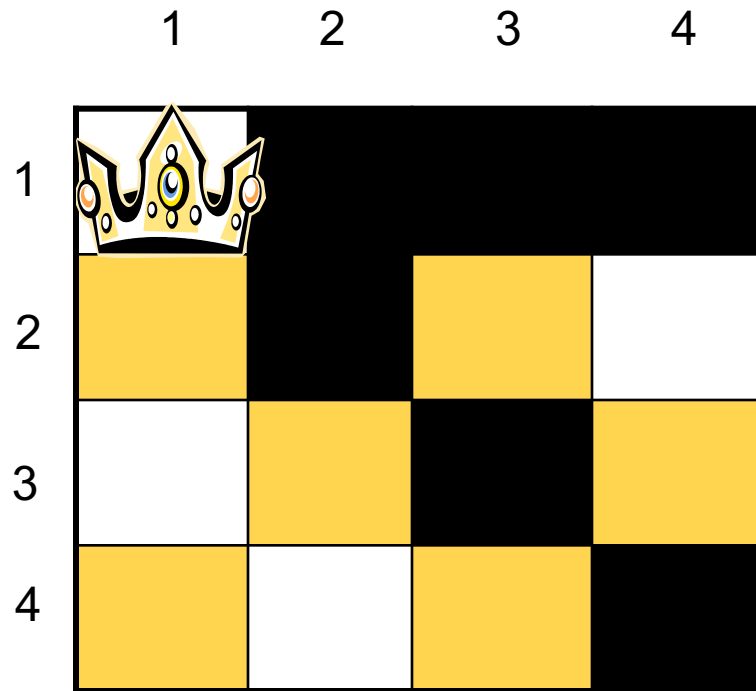
WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	<del>RB</del>	B	<del>B</del>	RGB

# Example: 4 Queens

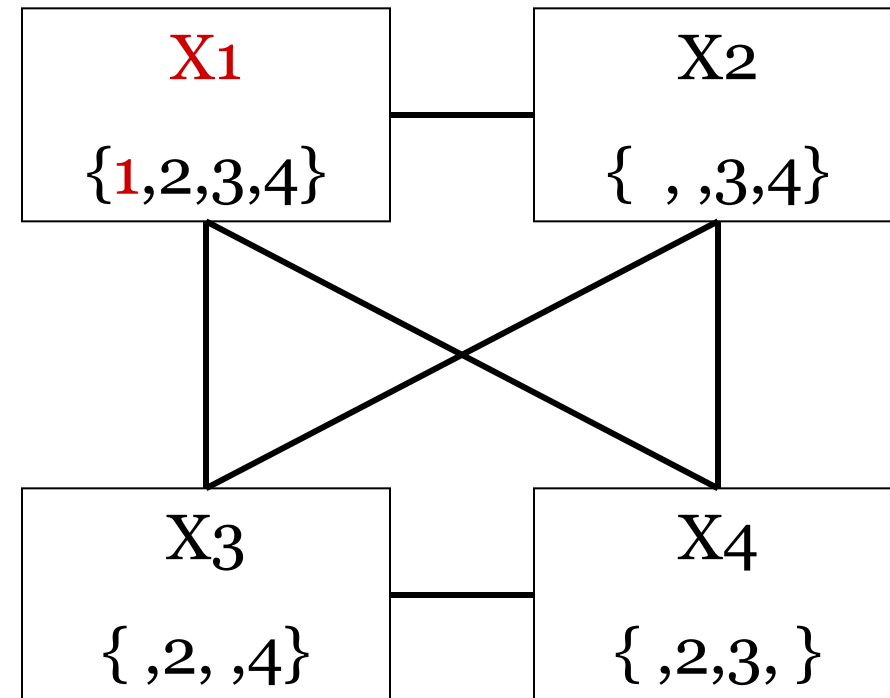
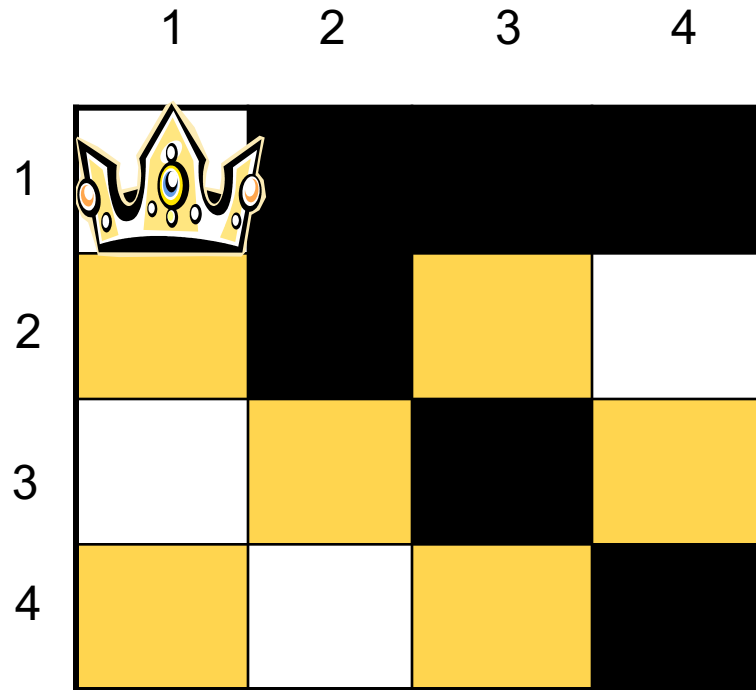
	1	2	3	4
1				
2				
3				
4				



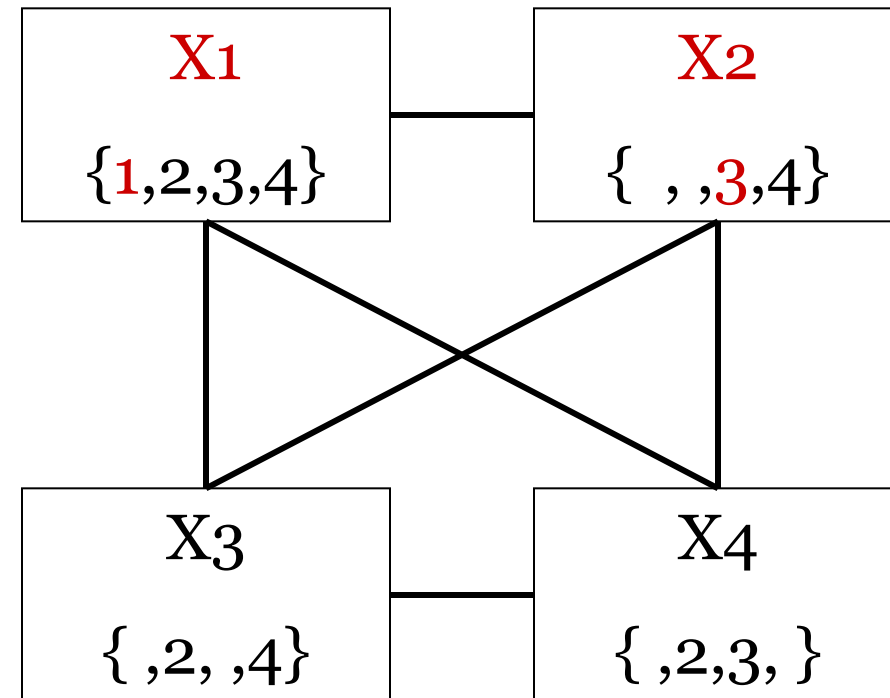
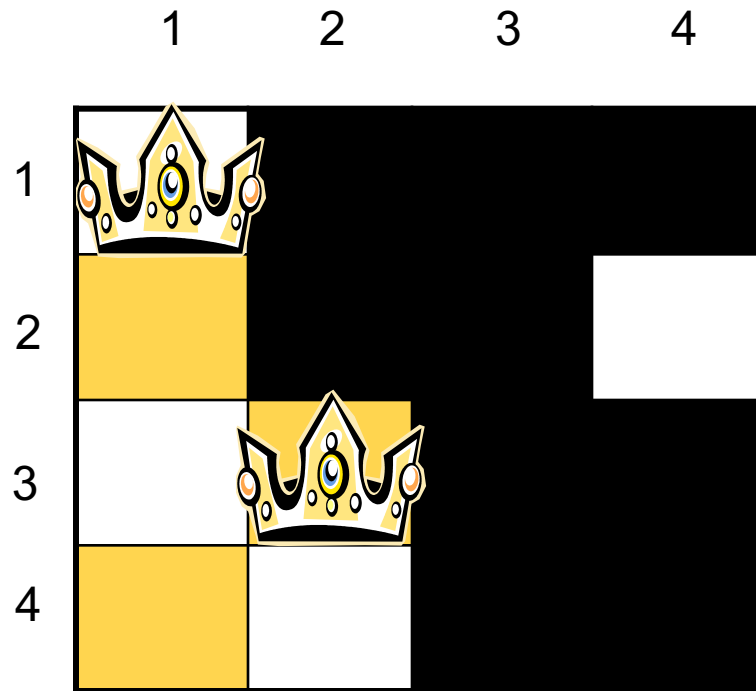
# Example: 4 Queens



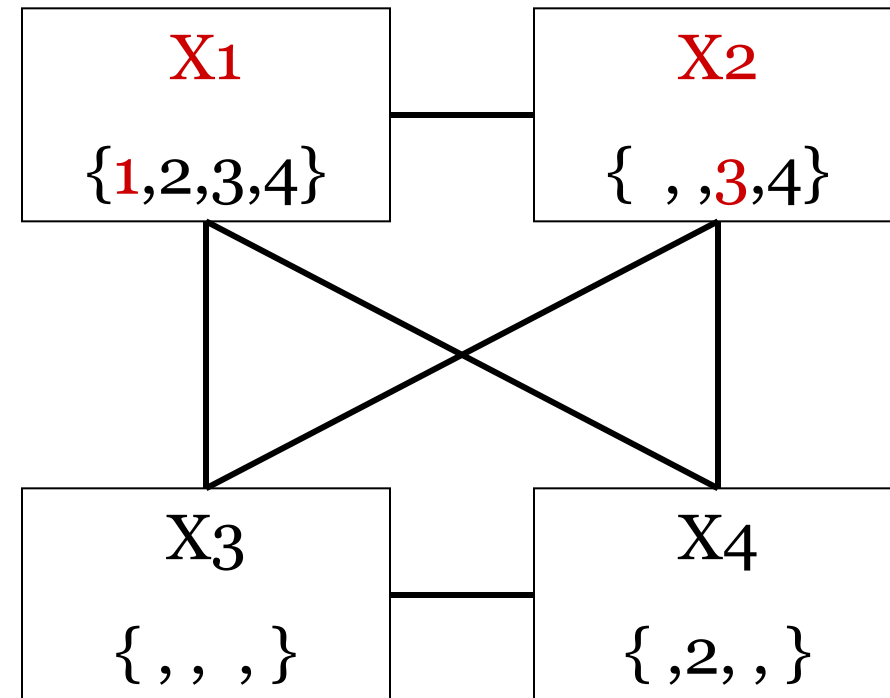
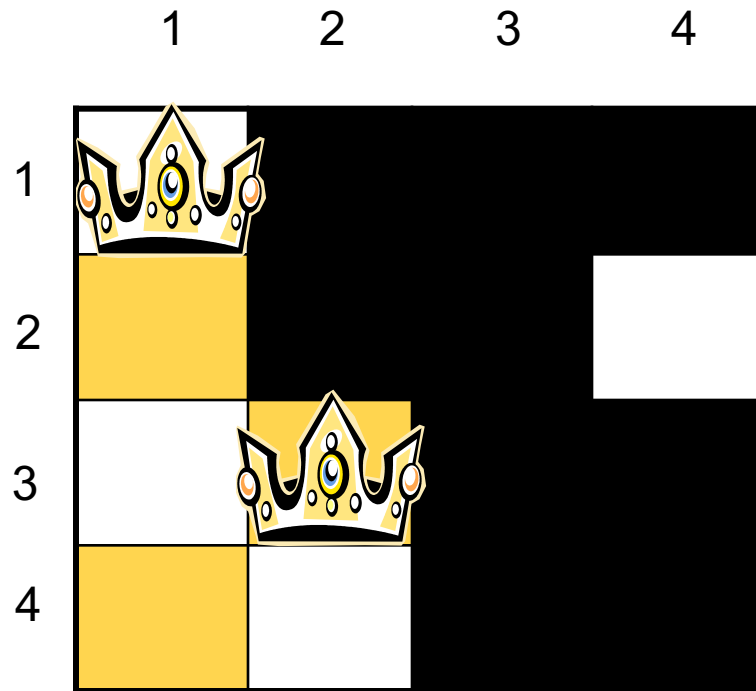
# Example: 4 Queens



# Example: 4 Queens



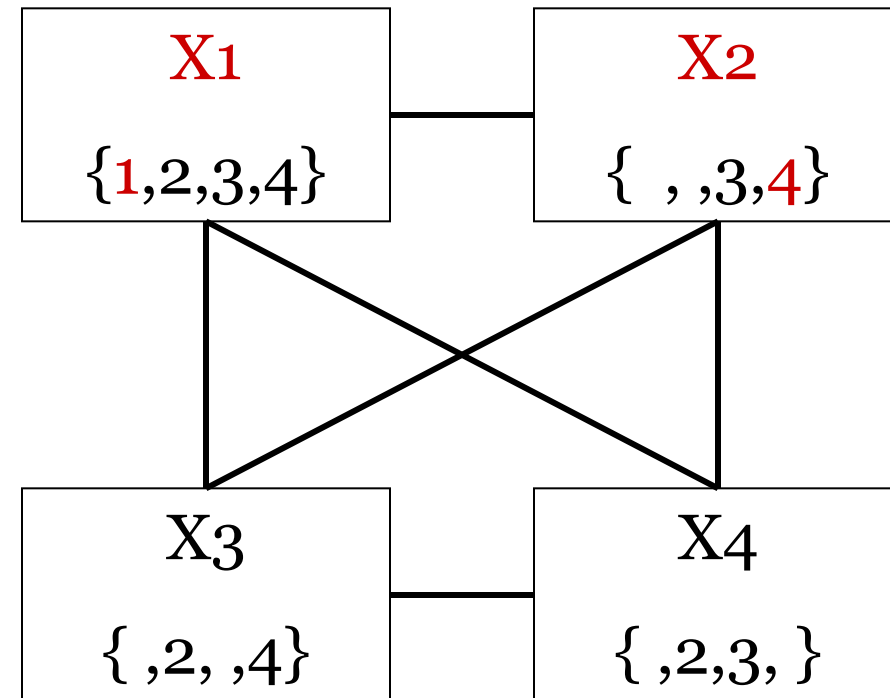
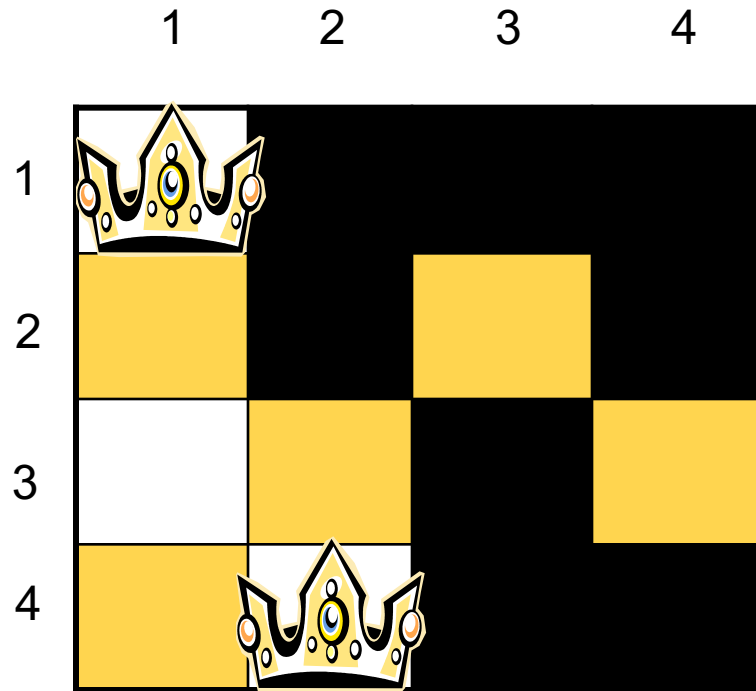
# Example: 4 Queens



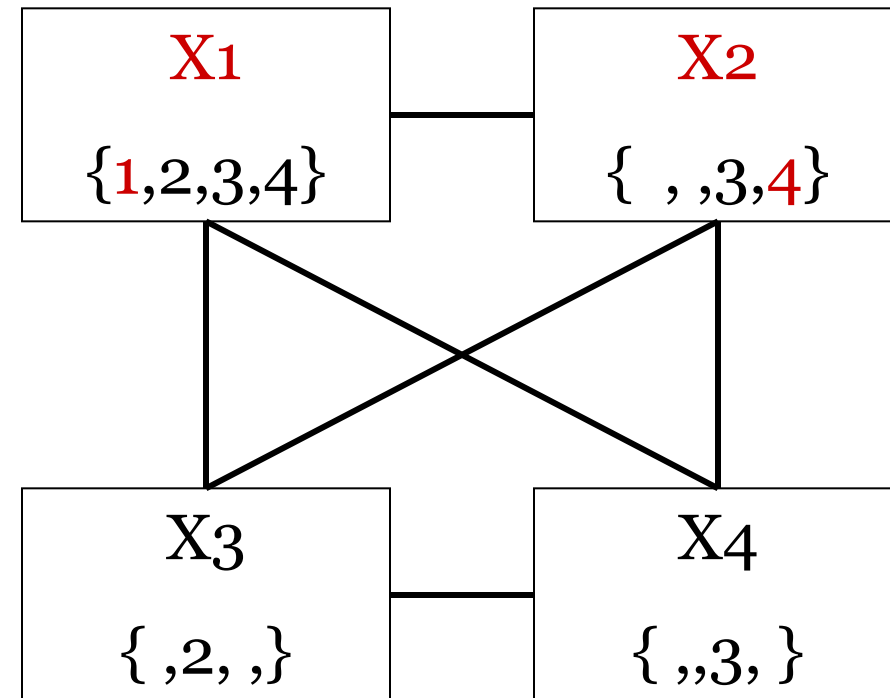
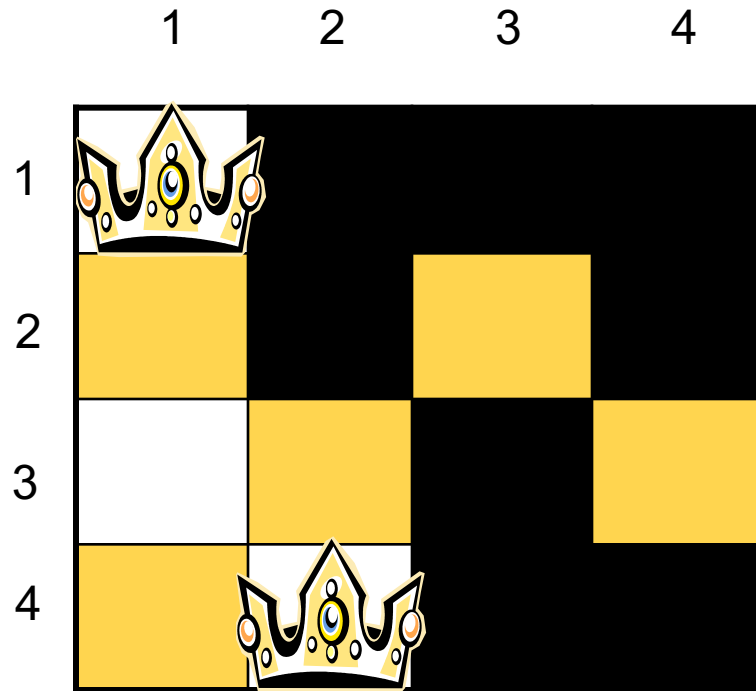
No possibilities for  $X_3$ , backtrack trying different value for  $X_2$



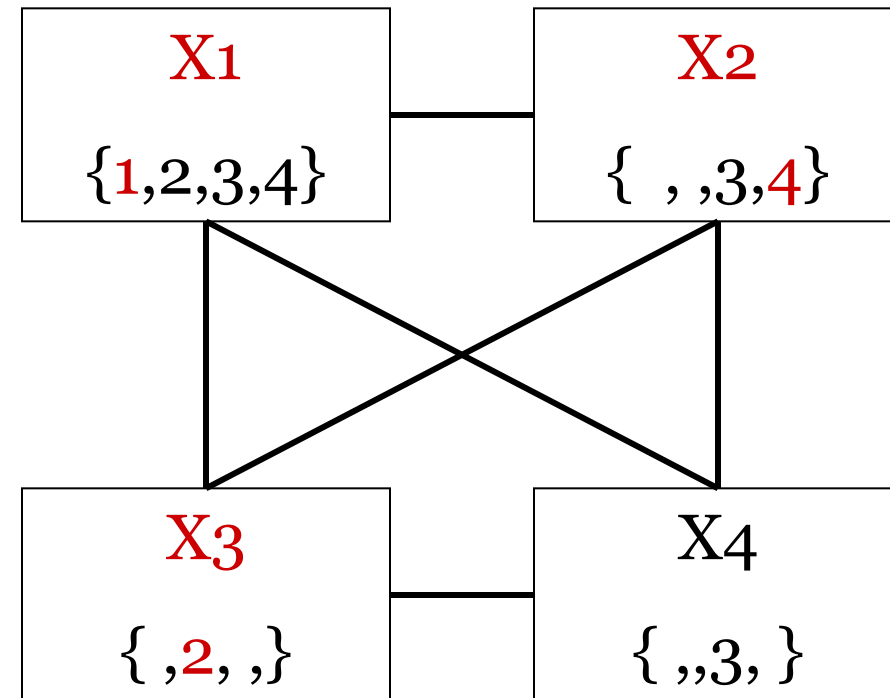
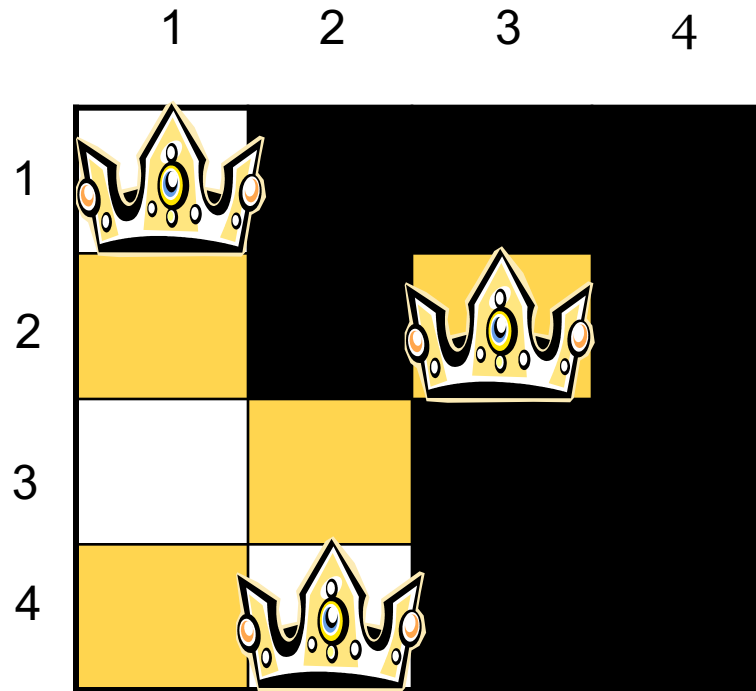
# Example: 4 Queens



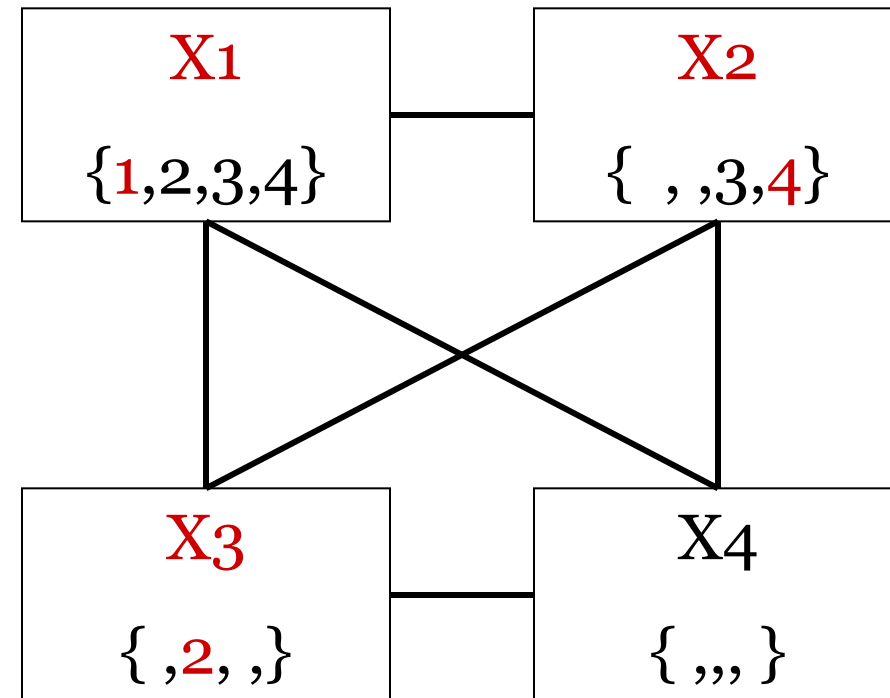
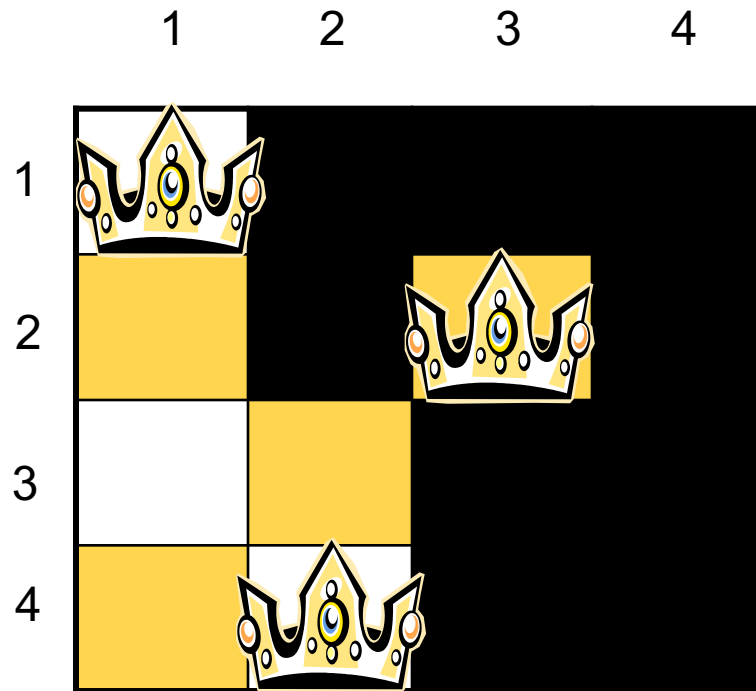
# Example: 4 Queens



# Example: 4 Queens

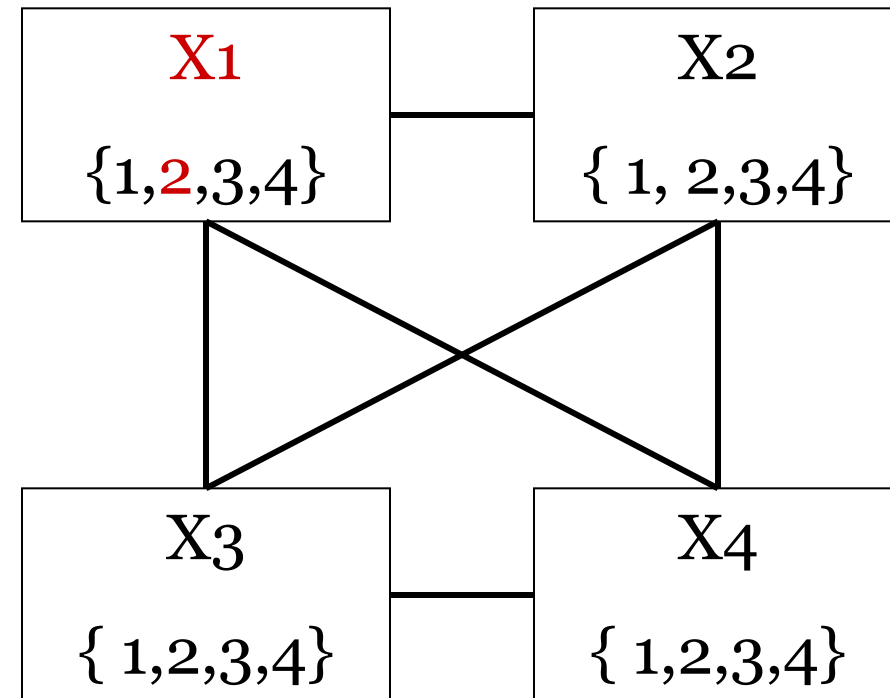
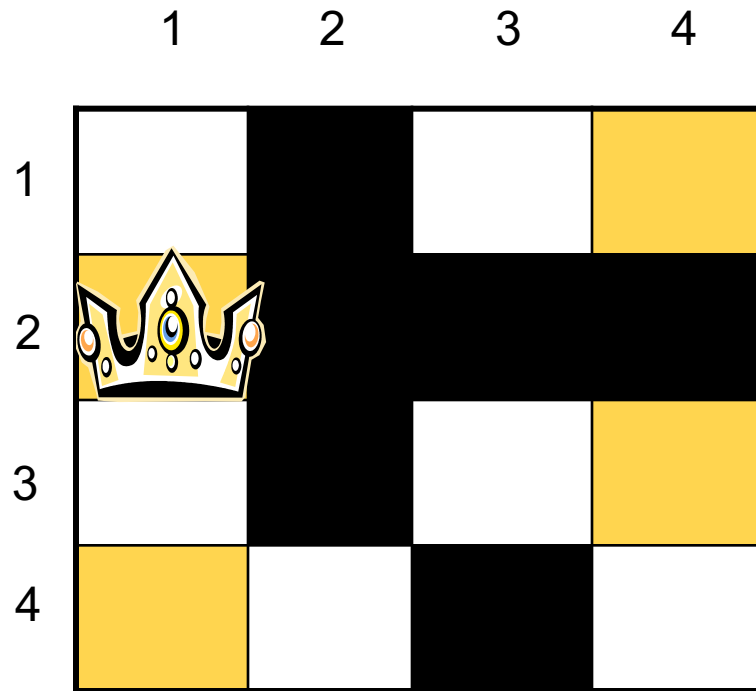


# Example: 4 Queens

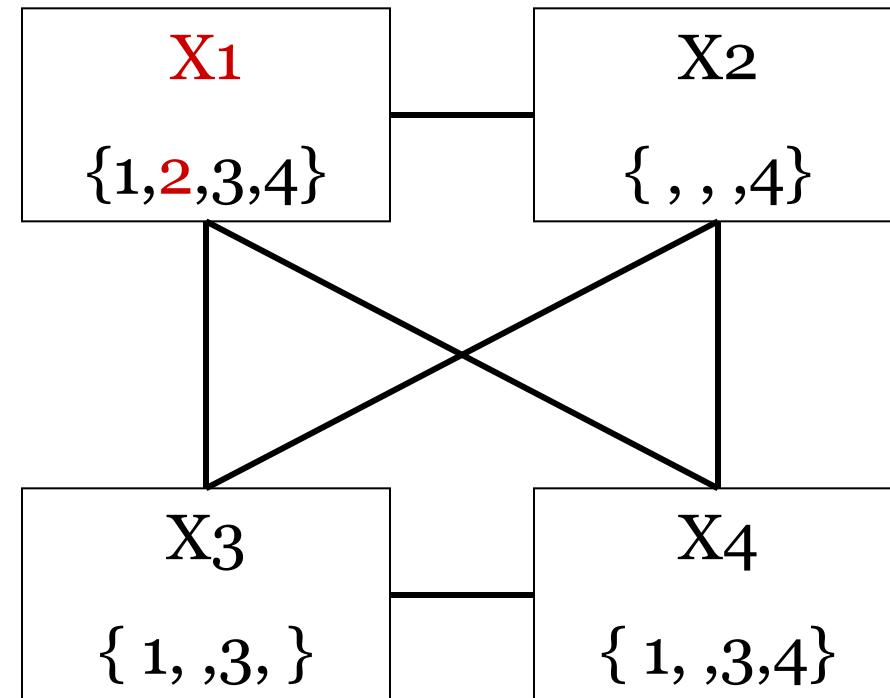
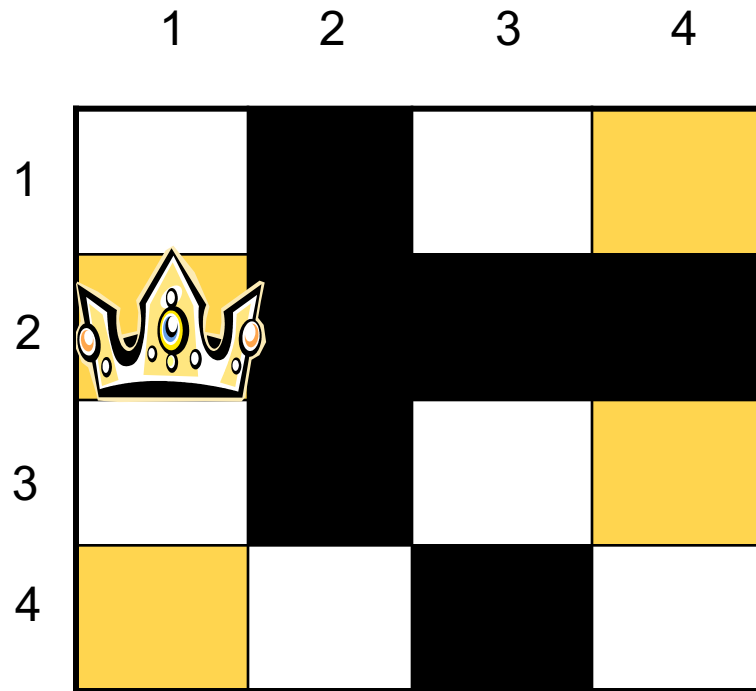


No possibilities for  $X_4$ , backtrack trying different value for  $X_1$

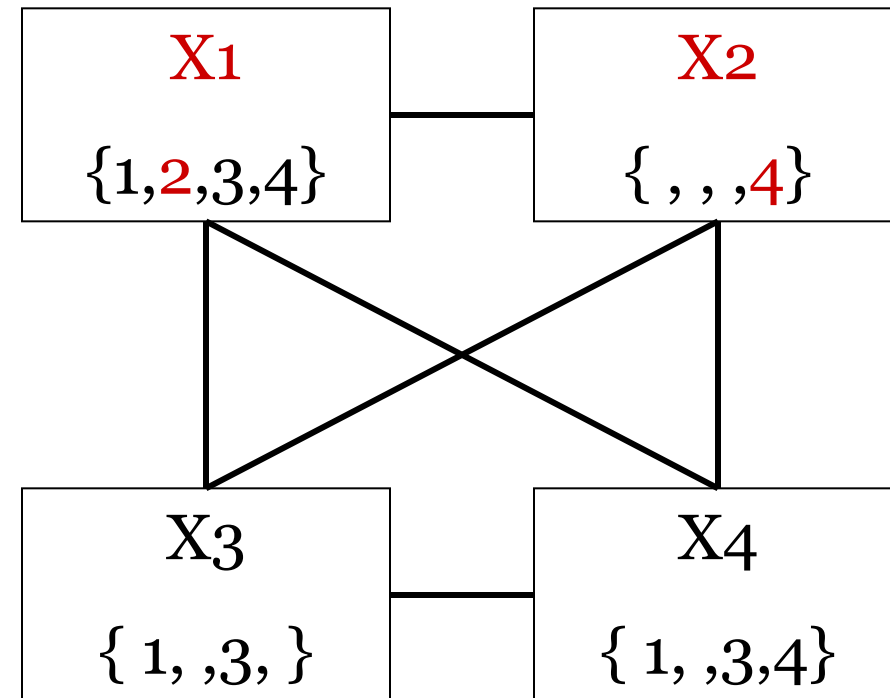
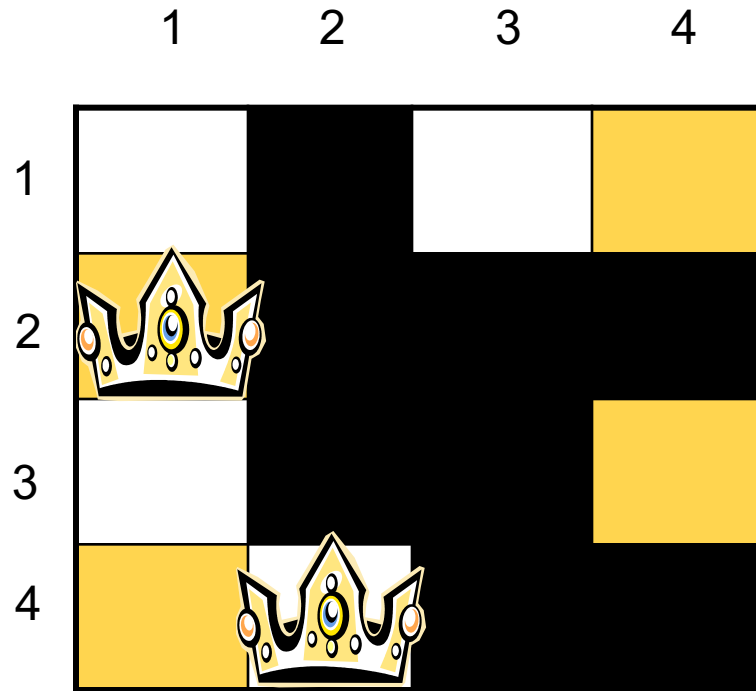
# Example: 4 Queens



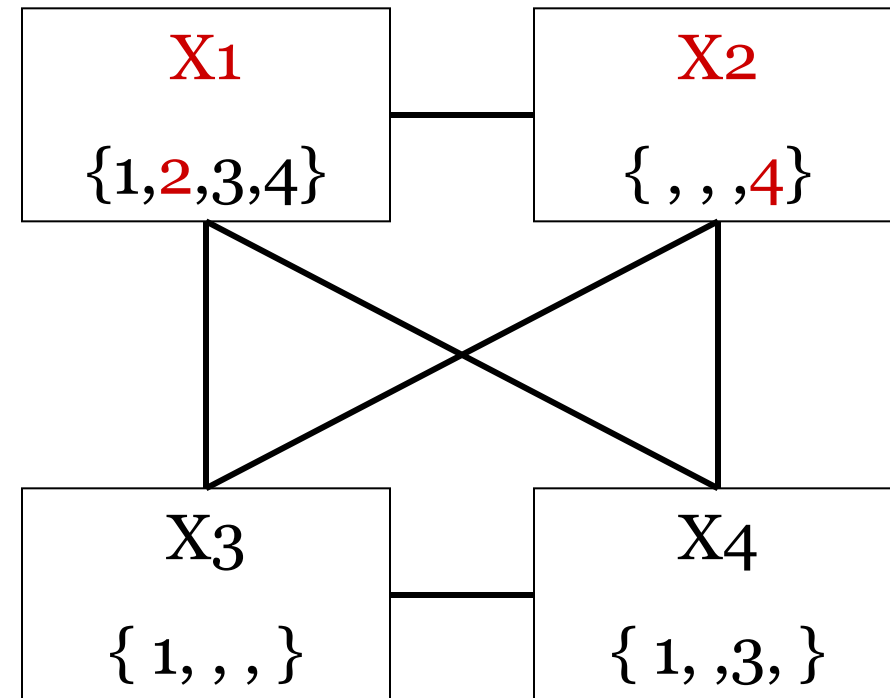
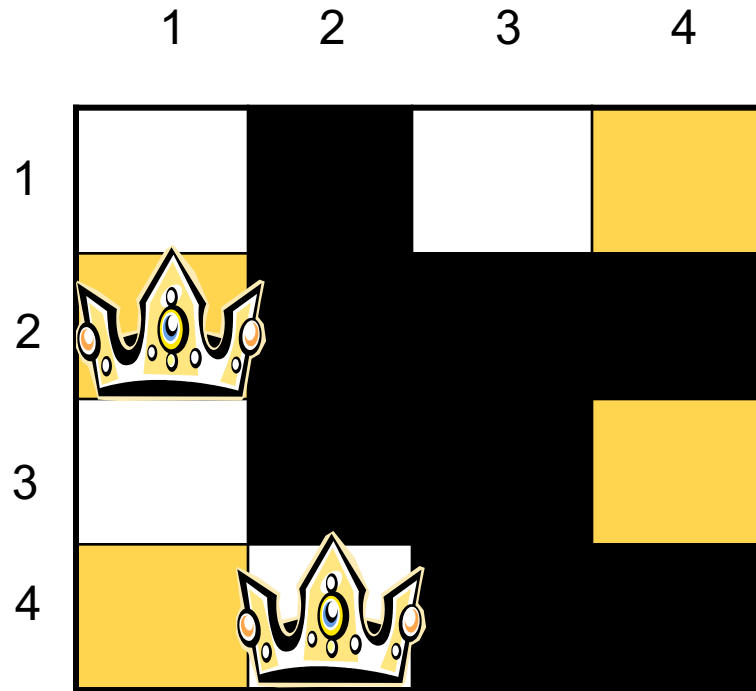
# Example: 4 Queens



# Example: 4 Queens

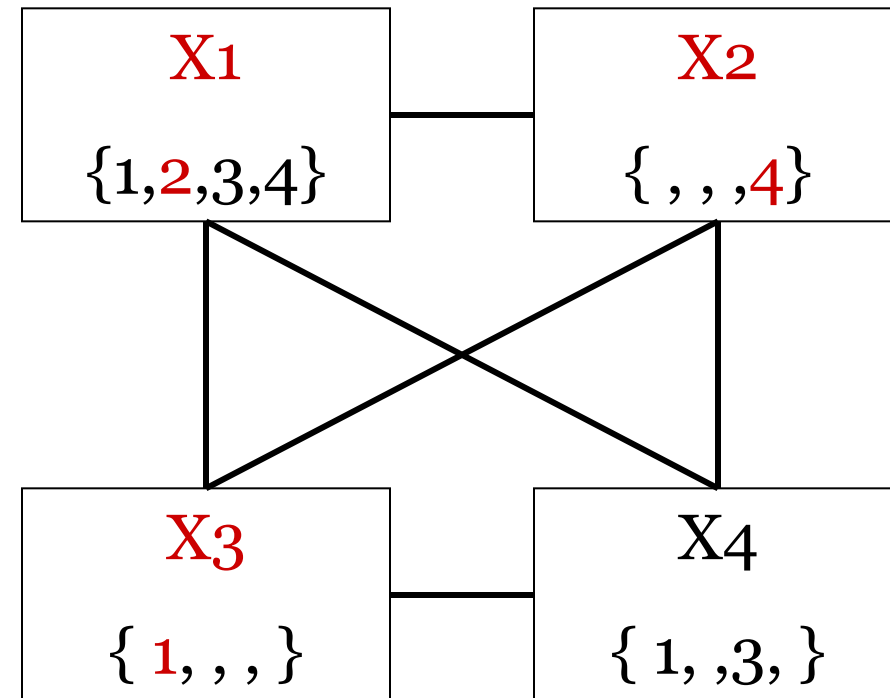
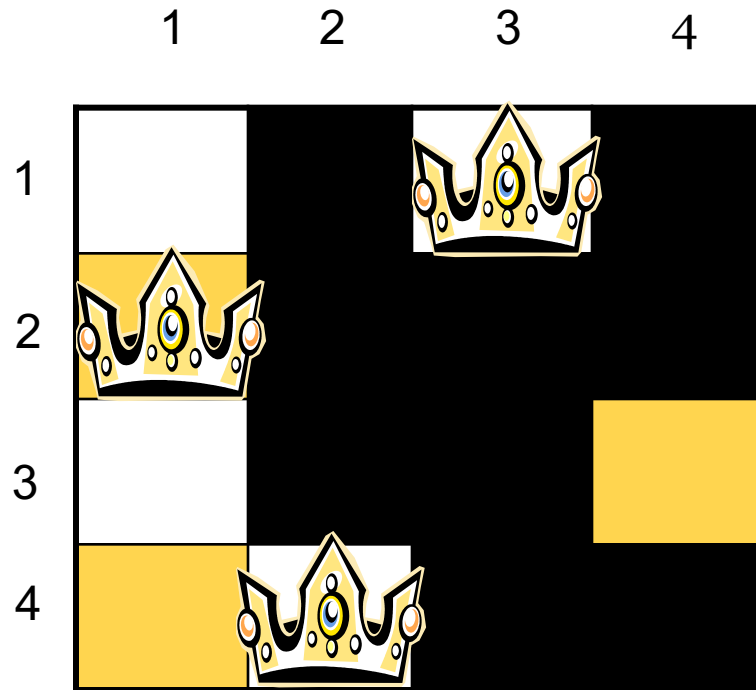


# Example: 4 Queens

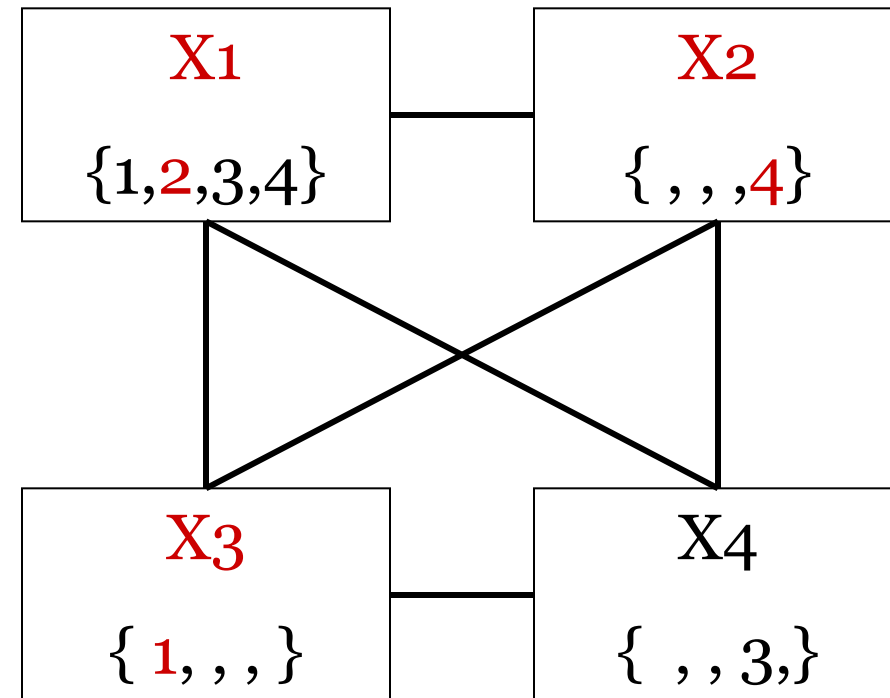
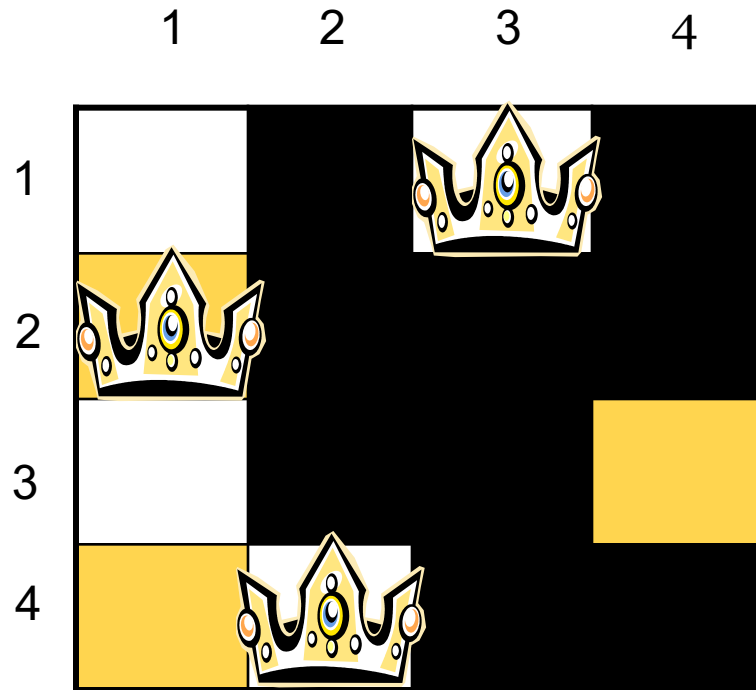




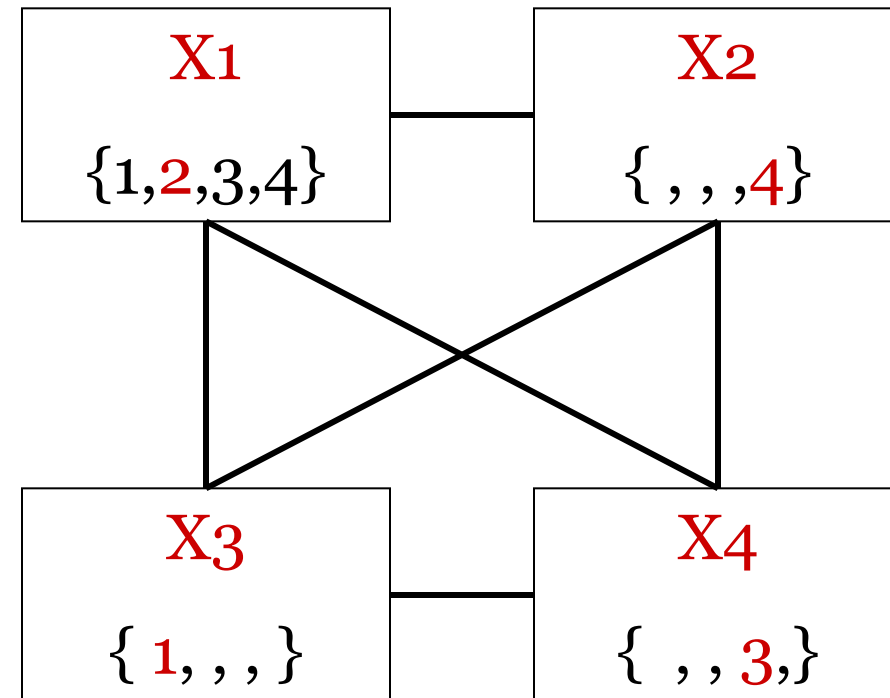
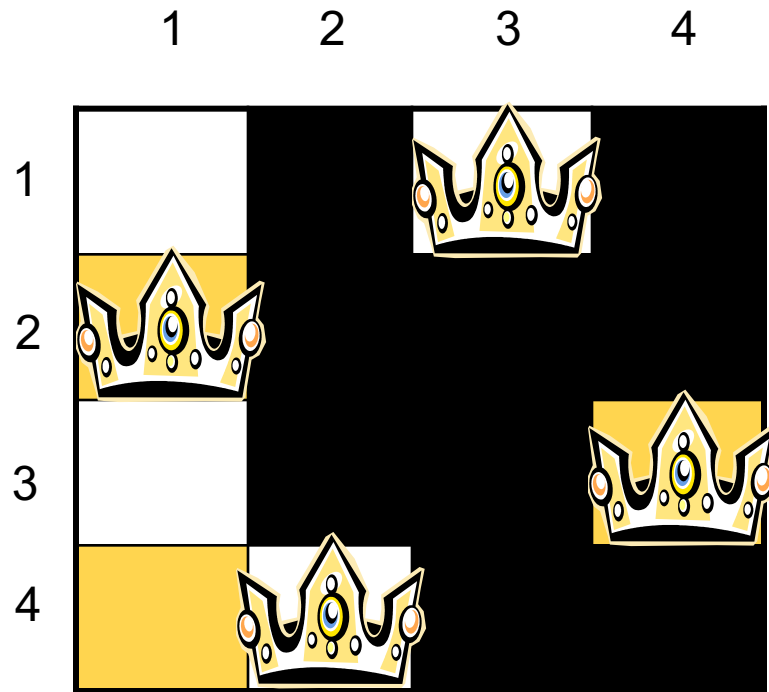
# Example: 4 Queens



# Example: 4 Queens



# Example: 4 Queens



# Summary

- What you should know
  - How to formalize problems as CSPs
  - Backtracking search
  - Heuristics
    - Variable ordering
    - Value ordering
  - Forward checking