

Lecture 21: Game Theory

CS486/686 Intro to Artificial Intelligence

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Outline

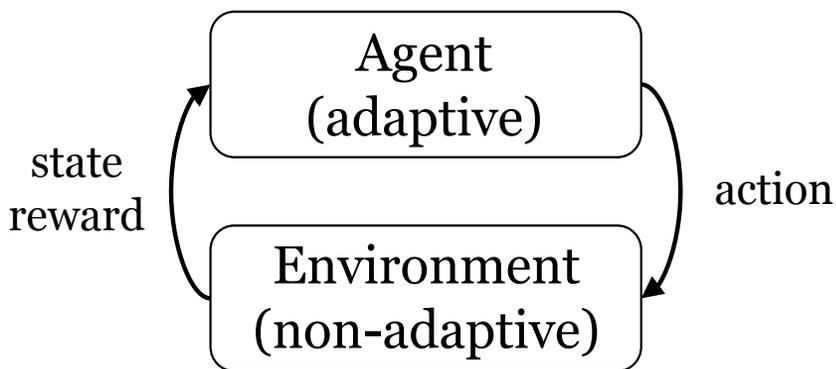
- Game Theory
- Normal form games
 - Strictly dominated strategies
 - Pure strategy Nash equilibria
 - Mixed Nash equilibria

Multi-agent Decision Making

- Sequential Decision Making
 - Markov Decision Processes
 - Reinforcement Learning
 - Multi-Armed Bandits
- All in single agent environments
- Real world environments: usually **more than one agent?**
 - **Each agent needs to account for other agents' actions/behaviours**

Reinforcement Learning

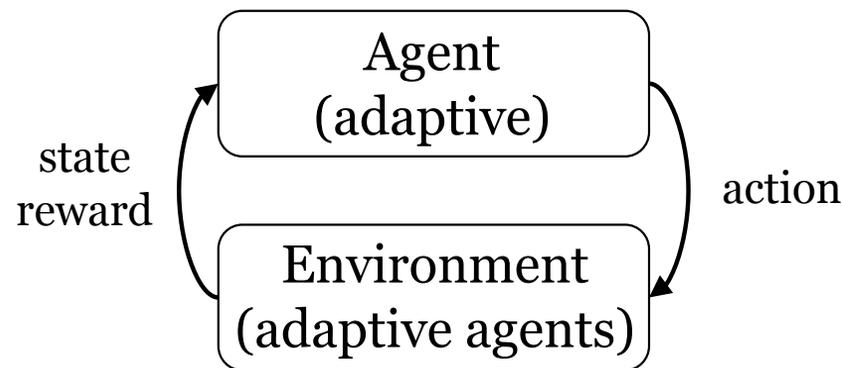
Single agent



Assumption: stationary transition function

$$P_t(s'|s, a) = P_{t'}(s'|s, a) \forall t, t'$$

Multiple agents



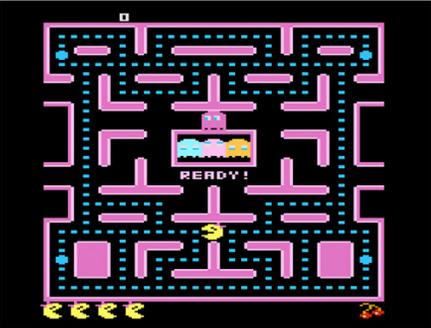
Non-stationary transition function

$$P_t(s'|s, a) \neq P_{t'}(s'|s, a)$$

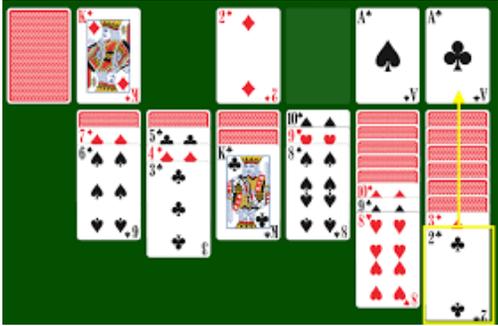
Game

- **Game:** Any scenario where outcomes depend on actions of **two or more rational and self-interested players**
 - **Players** (Decision Makers)
 - Agents within the game (observe states and take actions)
 - **Rational**
 - Agents choose their best actions (unless exploring)
 - **Self-interested**
 - Only care about their own benefits
 - May/May not harm others

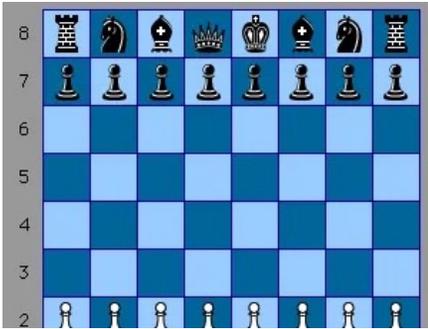
Which of these are games?



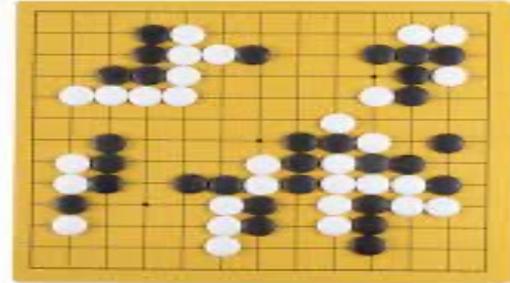
Atari



Solitaire



Chess



Go

Game Theory

- **Game Theory:** Mathematical model of **strategic interactions** among **multiple** rational agents in a game
 - **Interaction:**
 - One agent directly affects other agent(s)
 - Reward for one agent depends on other agent(s)
 - **Strategic:**
 - Agents **maximize their reward** by taking into account their influence (through actions) on the game
 - **Multiple:**
 - At-least two agents

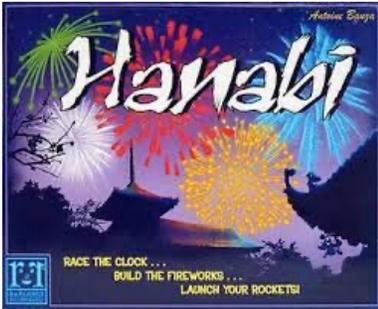
Game Theory Applications

- Auctions
- Diplomacy
- Negotiations
- Sports analytics
- Autonomous Driving
- Conversational agents

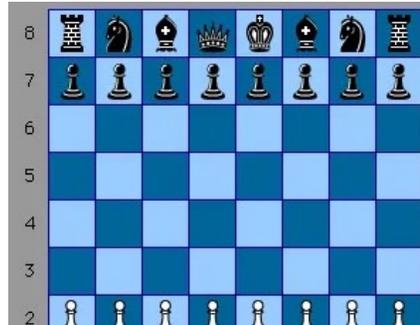
Categorization of Games

- Games can be
 - Cooperative:** agents have a common goal
 - Competitive:** agents have conflicting goals
 - Mixed:** in between cooperative and competitive (agents have different goals, but they are not conflicting)

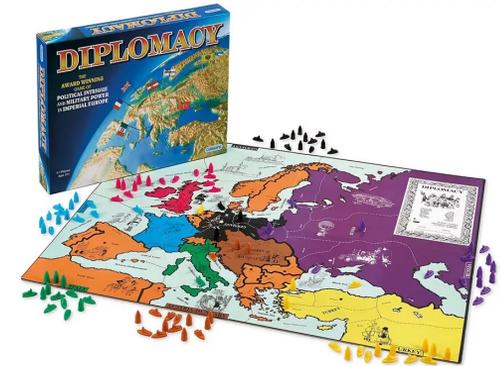
Cooperative



Competitive



Mixed



Normal Form Games

- Set of **agents**: $I = 1, 2, \dots, N$, where $N \geq 2$
- Set of **actions** for each agent: $A_i = \{a_i^1, \dots, a_i^m\}$
 - Game outcome is a **strategy profile (joint action)**: $a = (a_1, \dots, a_n)$
 - Total space of joint actions: $a \in \{A_1 \times A_2 \times \dots \times A_N\}$
- **Reward function** for each agent: $R_i: A \rightarrow \mathfrak{R}$, where $A = \{A_1 \times A_2 \times \dots \times A_N\}$
- **No state**
- **Horizon: $h = 1$**

Example: Even or Odd

		Agent 2	
		One	Two
Agent 1	One	2, -2	-3, 3
	Two	-3, 3	4, -4

Zero-sum game:

$$\sum_{i=1}^n R_i(a_1, \dots, a_n) = 0$$

$$I = \{1, 2\}$$

$$A_i = \{One, Two\}$$

An outcome is (One, Two)

$$R_1(One, Two) = -3 \text{ and } R_2(One, Two) = 3$$

Examples of strategic games

Baseball or Soccer

	B	S
B	2,1	0,0
S	0,0	1,2



Coordination Game

Chicken

	Cross	Turn
Cross	-1,-1	10,0
Turn	0,10	5,5



Anti-Coordination Game

Example: Prisoner's Dilemma



Confess

Don't Confess

Confess

-5,-5

0,-10

Don't
Confess

-10,0

-1,-1

Playing a game

- We now know how to describe a game
- Next step – **Playing the game!**
- Recall, agents are **rational**
 - Let p_i be agent i 's beliefs about what its opponents will do
 - Agent i is rational if it chooses to play a_i^* where

$$a_i^* = \operatorname{argmax}_{a_i} \sum_{a_{-i}} R(a_i, a_{-i}) p_i(a_{-i})$$

Notation: $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$ *actions of all agents except agent i*

Dominated Strategies

- **Definition:** A strategy a_i is *strictly dominated* if

$$\exists a'_i, \forall a_{-i}, R_i(a_i, a_{-i}) < R_i(a'_i, a_{-i})$$

- A rational agent will never play a strictly dominated strategy!
 - This allows us to solve some games!

Example: Prisoner's Dilemma

	Confess	Don't Confess
Confess	-5,-5	0,-10
Don't Confess	-10,0	-1,-1

	Confess	Don't Confess
Confess	-5,-5	0,-10

	Confess
Confess	-5,-5

Equilibrium Outcome

Strict Dominance does not capture the whole picture

	A	B	C
A	0,4	4,0	5,3
B	4,0	0,4	5,3
C	3,5	3,5	6,6

What strict dominance eliminations can we do?

None

Nash Equilibrium

- Sometimes an agent's best-response depends on the strategies other agents are playing
- A strategy profile, \mathbf{a}^* , is a **Nash equilibrium** if no agent has incentive to deviate from its strategy *given that others do not deviate*:

$$\forall i, a_i, \quad R_i(a_i^*, a_{-i}^*) \geq R_i(a_i, a_{-i}^*)$$

Nash Equilibrium

- Equivalently, \mathbf{a}^* is a Nash equilibrium iff $\forall i a_i^* = \operatorname{argmax}_{a_i} R_i(a_i, \mathbf{a}_{-i}^*)$

	A	B	C
A	0,4	4,0	5,3
B	4,0	0,4	5,3
C	3,5	3,5	6,6

(C,C) is a Nash equilibrium because:

$$R_1(C, C) = \max\{R_1(A, C), R_1(B, C), R_1(C, C)\}$$

AND

$$R_2(C, C) = \max\{R_2(C, A), R_2(C, B), R_2(C, C)\}$$

Exercise 1

What are the Nash Equilibria?

	B	S
B	2,1	0,0
S	0,0	1,2



Nash equilibria : (B,B) and (S,S)

Exercise 2

What are the Nash Equilibria?

		Agent 2	
		One	Two
Agent 1	One	2, -2	-3, 3
	Two	-3, 3	4, -4

No Nash equilibrium

(Mixed) Nash Equilibria

- **Mixed strategy σ_i :** σ_i defines a probability distribution over A_i
- **Strategy profile:** $\sigma = (\sigma_1, \dots, \sigma_n)$
- **Expected utility:** $R_i(\sigma) = \sum_a (\prod_j \sigma(a_j)) R_i(a)$
- **Nash Equilibrium:** σ^* is a (mixed) Nash equilibrium if

$$\forall i R_i(\sigma_i^*, \sigma_{-i}^*) \geq R_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i'$$

Finding Mixed Nash Equilibria

- Two players: $\sigma = (\sigma_1, \sigma_2)$
 - Let $p = \sigma_1$ and $q = \sigma_2$
- At the equilibrium:
 - p^* should be best strategy given q^* : $p^* = \operatorname{argmax}_p R_1(p, q^*)$
 - q^* should be best strategy given p^* : $q^* = \operatorname{argmax}_q R_2(p^*, q)$
- Solve system of equations:
 - $\frac{\partial}{\partial p} R_1(p, q) = 0$
 - $\frac{\partial}{\partial q} R_2(p, q) = 0$

Exercise 2 Revisited

		One	Two
A	One	2,-2	-3,3
	Two	-3,3	4,-4

$$p = \Pr(\text{one})$$

$$q = \Pr(\text{one})$$

How do we determine p and q ?

$$R_A(p, q) = 2pq - 3p(1 - q) - 3(1 - p)q + 4(1 - p)(1 - q)$$

$$R_B(p, q) = -2pq + 3p(1 - q) + 3(1 - p)q - 4(1 - p)(1 - q)$$

$$\frac{\partial}{\partial p} R_A(p, q) = 12q - 7 \rightarrow q = \frac{7}{12}$$

$$\frac{\partial}{\partial q} R_B(p, q) = -12p + 7 \rightarrow p = \frac{7}{12}$$

Exercise 3

			2	
		B	S	
1	B	2,1	0,0	
	S	0,0	1,2	

This game has 3 Nash equilibria (2 pure strategy Nash equilibria and 1 mixed strategy Nash equilibrium). Find them.

$$R_1(p, q) = 2pq + 1(1-p)(1-q)$$

$$R_2(p, q) = 1pq + 2(1-p)(1-q)$$

$$\frac{\partial R_1}{\partial p} = 2q - (1-q) = 0 \implies q = \frac{1}{3}$$

$$\frac{\partial R_2}{\partial q} = p - 2(1-p) = 0 \implies p = \frac{2}{3}$$

Mixed Nash Equilibrium

- **Theorem** (Nash 1950):

Every game in which the strategy sets A_1, \dots, A_n have a finite number of elements has a mixed strategy equilibrium.

John Nash
Nobel Prize in Economics (1994)



Other Useful Theorems

- **Theorem:** In an n-player pure strategy game, if iterated elimination of strictly dominated strategies eliminates all but the strategies (a_1^*, \dots, a_n^*) then these strategies are the unique Nash equilibria of the game
- **Theorem:** Any Nash equilibrium will survive iterated elimination of strictly dominated strategies.

Nash Equilibrium

- Interpretations:
 - Focal points, self-enforcing agreements, stable social convention, consequence of rational inference..
- Criticisms
 - They may not be unique
 - Ways of overcoming this: Refinements of equilibrium concept, Mediation, Learning
 - They may be hard to find
 - People don't always behave based on what equilibria would predict