Lecture 15: Markov Decision Processes
CS486/686 Intro to Artificial Intelligence

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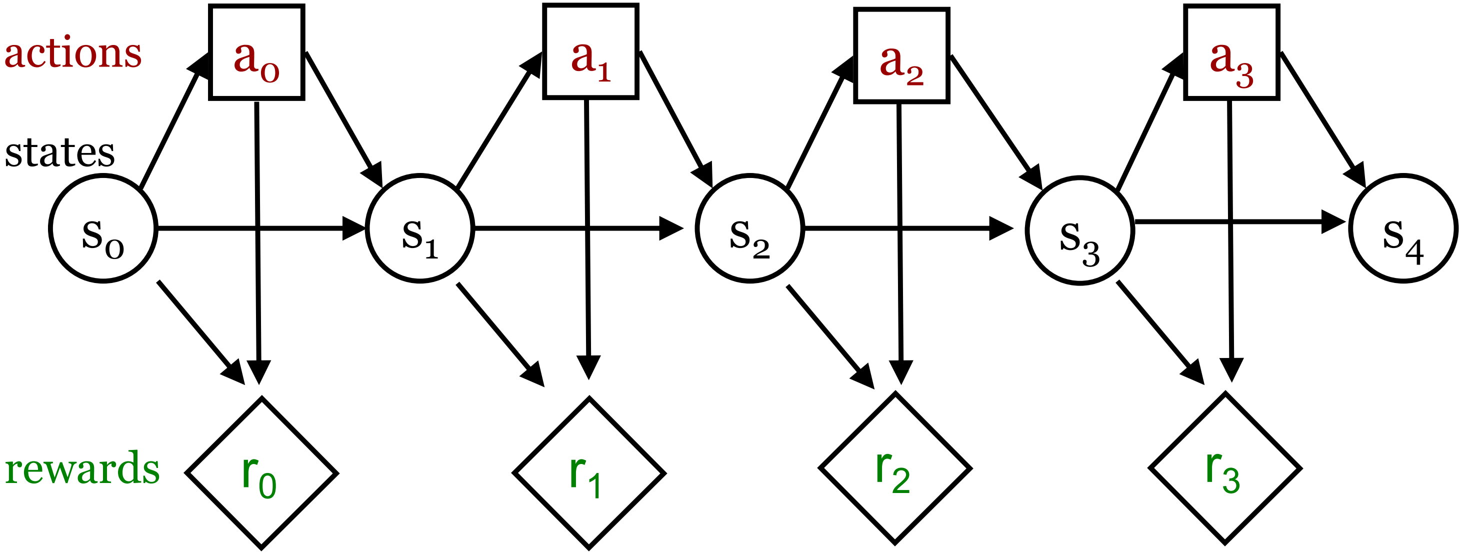
Outline

- Markov Decision Processes
    - Value Iteration

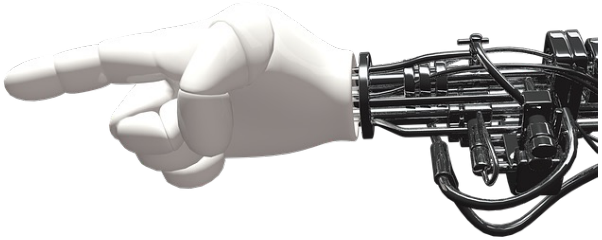
Markov Decision Processes

- Sequential decision process: how to choose actions to maximize rewards



Examples

- Robotic control
  - **States:** $\langle x, y, z, \theta \rangle$ coordinates of joints
  - **Actions:** forces applied to joints
  - **Rewards:** - distance to goal position



- Inventory management
  - **States:** inventory level
  - **Actions:** {doNothing, orderWidgets}
  - **Rewards:** sales - costs - storage

Markov Decision Process

Components	Formal definition	Inventory management
States	$s \in S$	inventory levels
Actions	$a \in A$	{doNothing, orderWidgets}
Rewards	$r \in \mathbb{R}$	Profit ($)
Transition model	$\Pr(s_t \mid s_{t-1}, a_{t-1})$	Stochastic demand
Reward model	$\Pr(r_t \mid s_t, a_t)$ $R(s_t, a_t) = \sum_{r_t} r_t \Pr(r_t \mid s_t, a_t)$	$R(s_t, a_t) =$ sales – costs – storage
Discount factor	$0 \leq \gamma \leq 1$	$\gamma = 0.999$
Horizon	$h \in \mathbb{N}$ or $h = \infty$	$h = \infty$

Common Assumptions

- Transition model
  - **Markovian**: $\Pr(s_{t+1}|s_t, a_t, s_{t-1}, a_{t-1}, \ldots) = \Pr(s_{t+1}|s_t, a_t)$
    - Current inventory and order sufficient to predict future inventory
  - **Stationary**: $\Pr(s_{t+1}|s_t, a_t)$ is same for all t
    - Distribution of demand same every day
- Reward model
  - **Stationary**: $R(s_t, a_t) = \sum_t r_t \Pr(r_t|s_t, a_t)$ is same for all t
    - Formula to compute profits is same every day
  - **Exception:** terminal reward is often different
    - In a game: 0 reward at each step and
      +1/-1 reward at the end for winning/losing

Discounted/Average Rewards

- Goal: maximize total rewards $\sum_{t=0}^{h} R(s_t, a_t)$
  Problem: if $h = \infty$, then $\sum_{t=0}^{h} R(s_t, a_t)$ may be infinite

- Solution 1: discounted rewards
  - Discount factor: $0 \leq \gamma < 1$
  - Finite utility: $\sum_t \gamma^t R(s_t, a_t)$ is a geometric sum
  - γ induces an inflation rate of $1/\gamma - 1$ (prefer utility sooner than later)

- Solution 2: average rewards
  - More complicated computationally (beyond scope of this course)

Policy

- Choice of action at each time step

- Formally:
  - Mapping from states to actions: $\pi(s_t) = a_t$
  - Assumption: fully observable states
    - Allows a_t to be chosen only based on current state s_t

Policy Optimization

- **Policy evaluation:** compute expected utility

$$V^{\pi}(s_0) = \sum_{t=0}^{h} \gamma^t \sum_{s_{t+1}} \Pr(s_{t+1}|s_0, \pi)\, R(s_{t+1}, \pi(s_{t+1}))$$

- **Optimal policy π^*:** policy with highest expected utility

$$V^{\pi^*}(s_0) \geq V^{\pi}(s_0) \;\; \forall \pi$$

- Several classes of algorithms:
  - Value iteration
  - Policy iteration
  - Linear Programming
  - Search techniques

Value Iteration

- Value when no time left:

$$V_0^*(s_h) = \max_{a_h} R(s_h, a_h)$$

- Value with one time step left:

$$V_1^*(s_{h-1}) = \max_{a_{h-1}} R(s_{h-1}, a_{h-1}) + \gamma \sum_{s_h} \Pr(s_h | s_{h-1}, a_{h-1})\, V_0^*(s_h)$$

- Value with two time steps left:

$$V_2^*(s_{h-2}) = \max_{a_{h-2}} R(s_{h-2}, a_{h-2}) + \gamma \sum_{s_{h-1}} \Pr(s_{h-1} | s_{h-2}, a_{h-2})\, V_1^*(s_{h-1})$$

- …
- Bellman's equation:

$$V_\infty^*(s_t) = \max_{a_t} R(s_t, a_t) + \gamma \sum_{s_{t+1}} \Pr(s_{t+1} | s_t, a_t)\, V_\infty^*(s_{t+1})$$

$$a_t^* = \underset{a_t}{\operatorname{argmax}}\, R(s_t, a_t) + \gamma \sum_{s_{t+1}} \Pr(s_{t+1} | s_t, a_t)\, V_\infty^*(s_{t+1})$$

Value Iteration

```
valueIteration(MDP)
    $V_0^*(s) \leftarrow \max_a R(s,a)\ \ \forall s$
    For $n = 1$ to $h$ do
        $V_n^*(s) \leftarrow \max_a R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_{n-1}^*(s')\ \ \forall s$
    Return $V^*$
```

Optimal policy π^*

$n = 0$: $\pi_0^*(s) \leftarrow \underset{a}{\operatorname{argmax}}\, R(s,a)\ \forall s$

$n > 0$: $\pi_n^*(s) \leftarrow \underset{a}{\operatorname{argmax}}\, R(s,a) + \gamma \sum_{s'} \Pr(s'|s,a)\, V_{n-1}^*(s')\ \ \forall s$

NB: π^* is non-stationary (i.e., time dependent)

Value Iteration (Matrix Form)

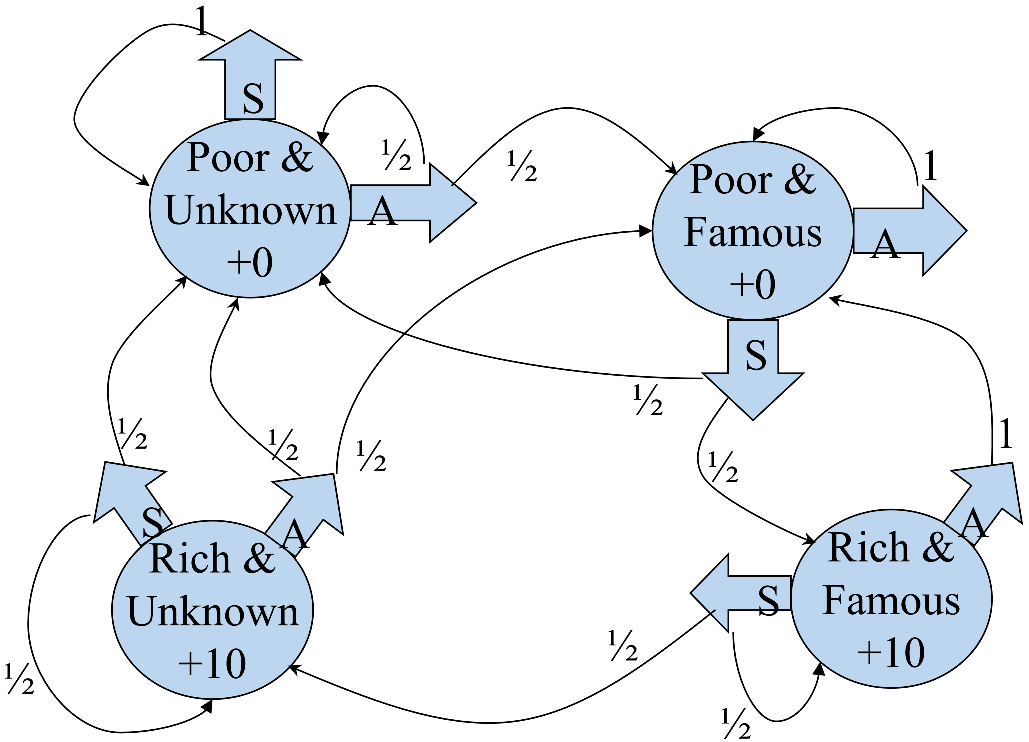
R^a: $|S|$×1 column vector of rewards for a

V_n^*: $|S|$×1 column vector of state values

T^a: $|S|$×$|S|$ matrix of transition probabilities for a

```
valueIteration(MDP)
  $V_0^* \leftarrow \max_a R^a$
  For $t = 1$ to $h$ do
      $V_n^* \leftarrow \max_a R^a + \gamma T^a V_{n-1}^*$
  Return $V^*$
```

A Markov Decision Process

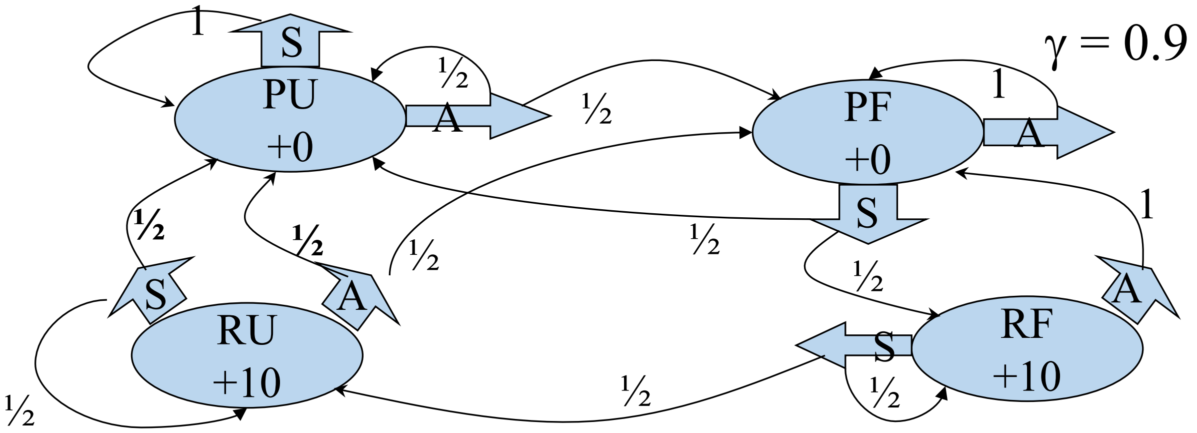


$\gamma = 0.9$

You own a company

In every state you must choose between

Saving money or **A**dvertising



n	$V^*_n(PU)$	$\pi^*_n(PU)$	$V^*_n(PF)$	$\pi^*_n(PF)$	$V^*_n(RU)$	$\pi^*_n(RU)$	$V^*_n(RF)$	$\pi^*_n(RF)$
0	0	A,S	0	A,S	10	A,S	10	A,S
1	0	A,S	4.5	S	14.5	S	19	S
2	2.03	A	8.55	S	16.53	S	25.08	S
3	4.76	A	12.20	S	18.35	S	28.72	S
4	7.63	A	15.07	S	20.40	S	31.18	S
5	10.21	A	17.46	S	22.61	S	33.21	S

Exercise: Value Iteration, No Time Left (RF State)

$$V_0(RF) = \max\{R(RF, A), R(RF, S)\}$$
$$= \max\{10, 10\}$$
$$= 10$$

$$\pi_0(RF) = \operatorname{argmax}\{R(RF, A), R(RF, S)\}$$
$$= \{A, S\}$$

Exercise: Value Iteration, One Time Step Left (RF State)

$$V_1(RF) = \max_a R(RF,a) + \gamma \sum_{s'} P(s'|RF,a) V_0(s')$$

$$= \max\{10 + 0.9(1)(0), \quad 10 + 0.9(0.5 \times 10 + 0.5 \times 10)\}$$

$$= \max\{10, 19\}$$

$$= 19$$

$$\pi_1(RF) = \text{argmax}\{10, 19\}$$

$$= S$$

Horizon Effect

- Finite h:
  - Non-stationary optimal policy
  - Best action different at each time step
  - Intuition: best action varies with the amount of time left

- Infinite h:
  - Stationary optimal policy
  - Same best action at each time step
  - Intuition: same (infinite) amount of time left at each time step
  - **Problem**: value iteration does infinite # of iterations

Infinite Horizon

- Assuming a discount factor γ, after n time steps, rewards are scaled down by γ^n
- For large enough n, rewards become insignificant since $\gamma^n \rightarrow 0$

- Solution #1:
  - pick large enough n and run value iteration for n steps
  - Execute policy π_n found at the n^{th} iteration

$\|\vec{x}\|_\infty = \max |x_i|$

- Solution #2:
  - Continue iterating until $||V_n - V_{n-1}||_\infty \leq \epsilon$ (ϵ is called tolerance)
  - Execute policy π_n found at the n^{th} iteration