

Lecture 11: Neural Networks

CS486/686 Intro to Artificial Intelligence

2026-2-10

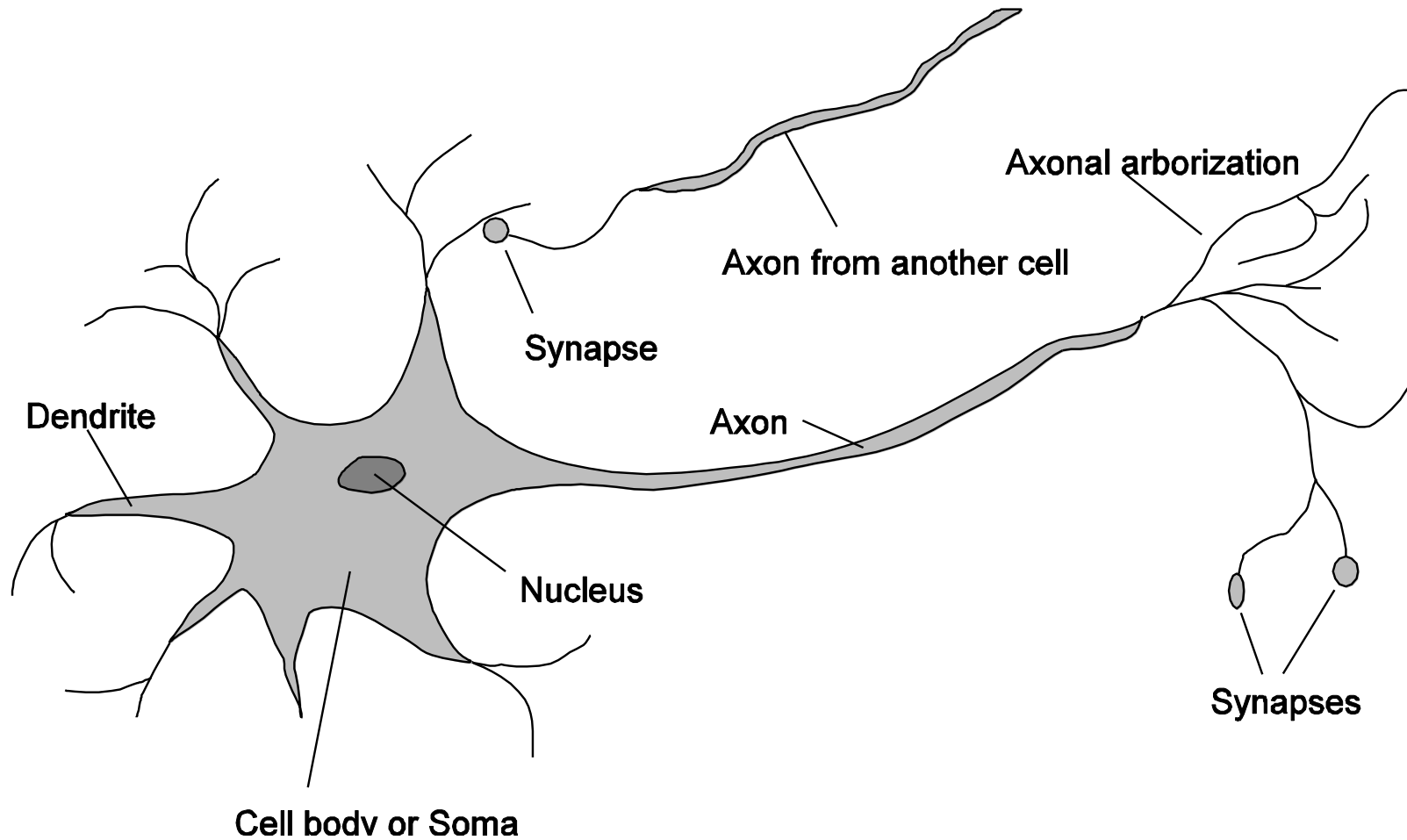
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Outline

- Neural networks
 - Perceptron
 - Supervised learning algorithms for neural networks

Neuron



Artificial Neural Networks

- Idea: **mimic the brain to do computation**
- Artificial neural network:
 - Nodes (a.k.a. units) correspond to neurons
 - Links correspond to synapses
- Computation:
 - Numerical signal transmitted between nodes corresponds to chemical signals between neurons
 - Nodes modifying numerical signal corresponds to neurons firing rate

ANN Unit

For each unit i :

- **Weights: W**

- Strength of the link from unit i to unit j
- Input signals x_i weighted by W_{ji} and linearly combined:

$$a_j = \sum_i W_{ji} x_i + W_{j0} = \mathbf{W}_j \bar{\mathbf{x}}$$

- **Activation function: h**

- Numerical signal produced: $y_j = h(a_j)$

ANN Unit

- Picture

Activation Function

- Should be nonlinear
 - Otherwise, network is just a linear function
- Often chosen to mimic firing in neurons
 - Unit should be “active” (output near 1) when fed with the “right” inputs
 - Unit should be “inactive” (output near 0) when fed with the “wrong” inputs

Common Activation Functions

Threshold

Sigmoid

Logic Gates

- McCulloch and Pitts (1943)
 - Design ANNs to represent Boolean functions
- What should be the weights of the following units to code AND, OR, NOT ?

AND

OR

NOT

Network Structures

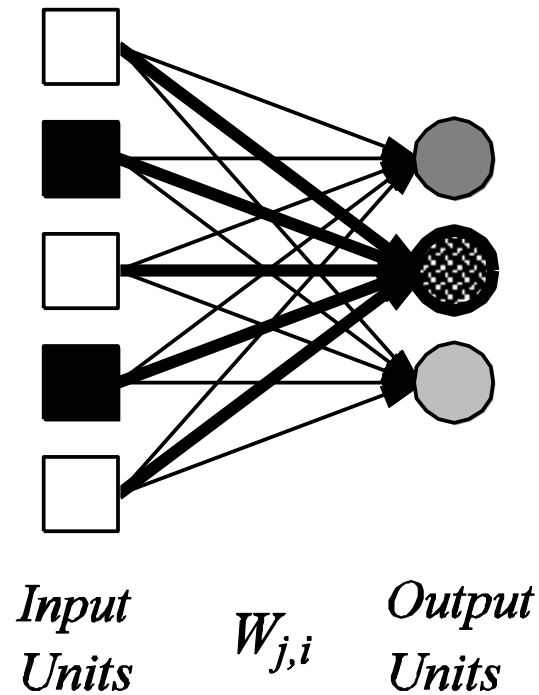
- **Feed-forward network**
 - Directed **acyclic** graph
 - No internal state
 - Simply computes outputs from inputs
- **Recurrent network**
 - Directed **cyclic** graph
 - Dynamical system with internal states
 - Can memorize information

Feed-forward network

- Simple network with two inputs, one hidden layer of two units, one output unit

Perceptron

- Single layer feed-forward network

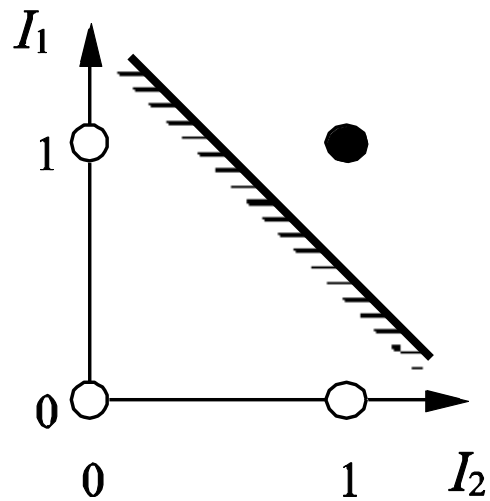


Threshold Perceptron Hypothesis Space

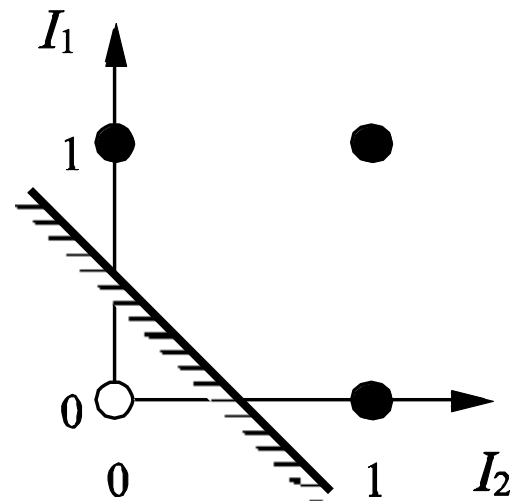
- Hypothesis space h_w :
 - All binary classifications with parameters w s.t.
$$w^T \bar{x} > 0 \rightarrow +1$$
$$w^T \bar{x} < 0 \rightarrow -1$$
- Since $w^T \bar{x}$ is linear in w , perceptron is called a **linear separator**

Linear Separability

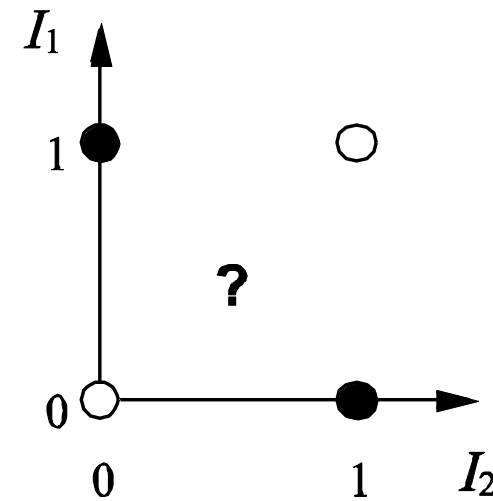
- Are all Boolean gates linearly separable?



(a) I_1 **and** I_2



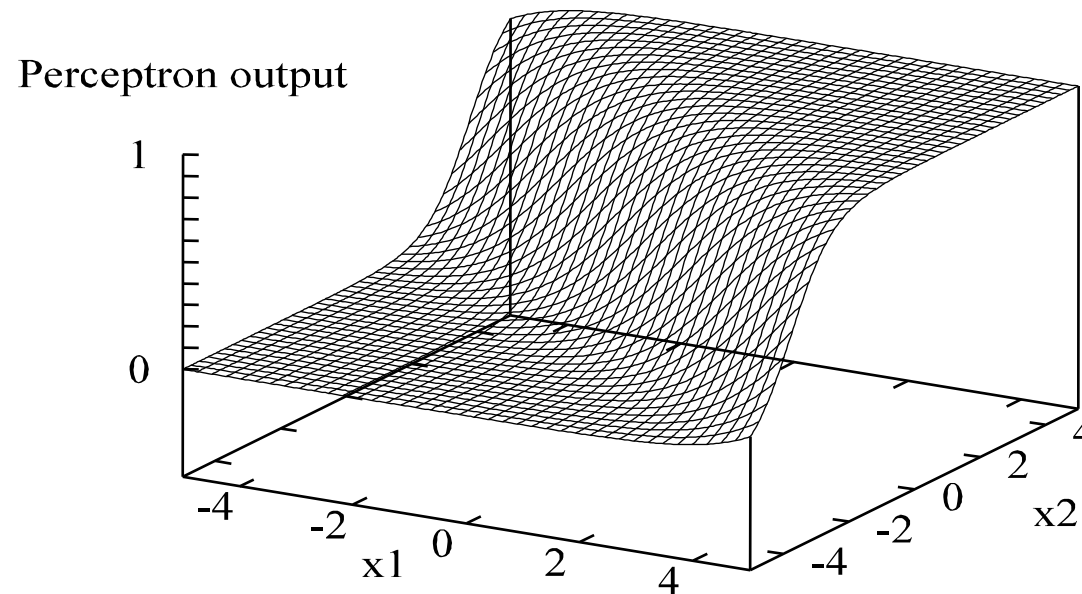
(b) I_1 **or** I_2



(c) I_1 **xor** I_2

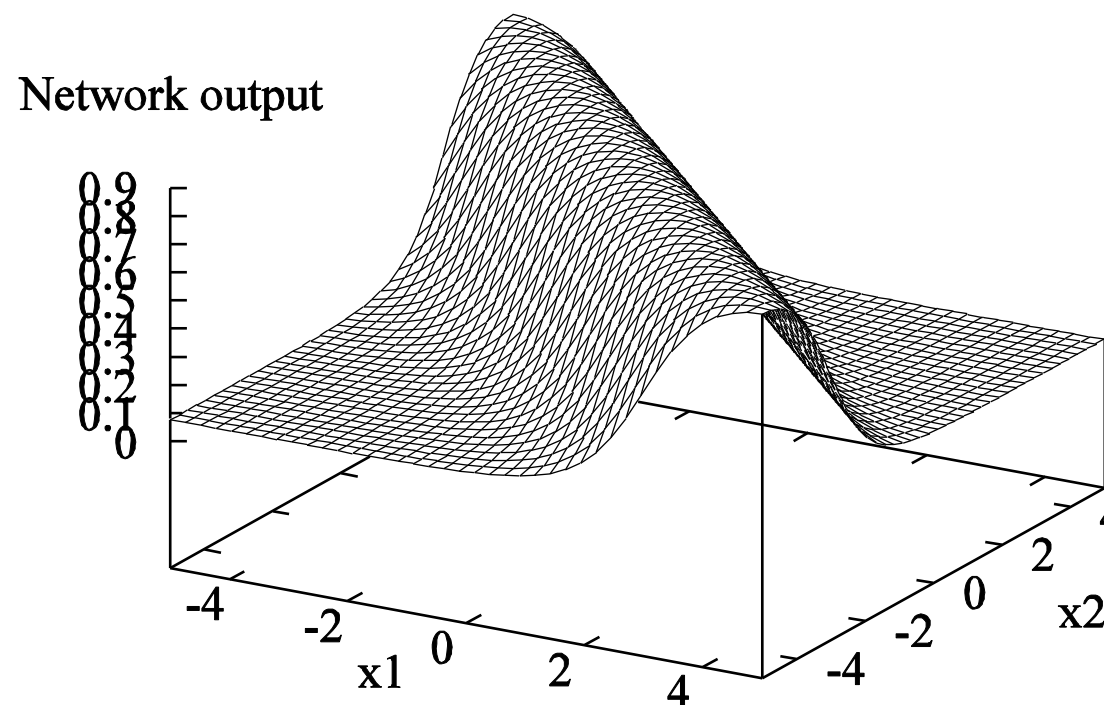
Sigmoid Perceptron

- Represent “soft” linear separators



Multilayer Networks

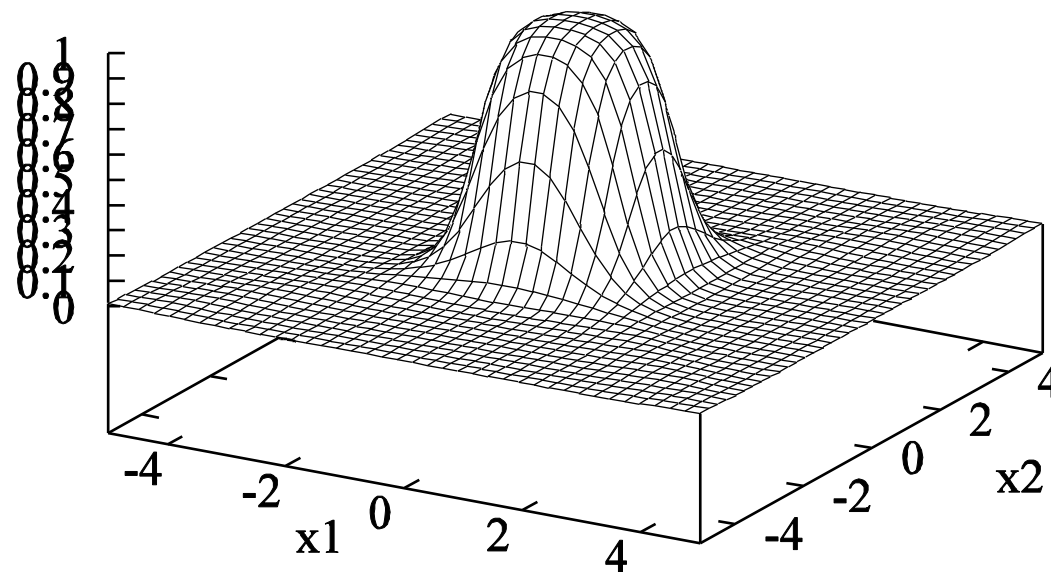
- Adding two sigmoid units with parallel but opposite “cliffs” produces a ridge



Multilayer Networks

- Adding two intersecting ridges (and thresholding) produces a bump

Network output



Multilayer Networks

- By tiling bumps of various heights together, we can approximate any function.
- **Theorem:** Neural networks with at least one hidden layer of sufficiently many sigmoid units can approximate any function arbitrarily closely.

Common activation functions h

- Threshold: $h(a) = \begin{cases} 1 & a \geq 0 \\ -1 & a < 0 \end{cases}$
- Sigmoid: $h(a) = \sigma(a) = \frac{1}{1+e^{-a}}$
- Gaussian: $h(a) = e^{-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2}$
- Tanh: $h(a) = \tanh(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}}$
- Identity: $h(a) = a$

Weight training

- Parameters: $\langle W^{(1)}, W^{(2)}, \dots \rangle$
- Objectives:
 - **Error minimization**
 - Backpropagation (aka “backprop”)
 - Maximum likelihood
 - Maximum a posteriori
 - Bayesian learning

Least squared error

- Error function

$$E(\mathbf{W}) = \frac{1}{2} \sum_n E_n(\mathbf{W})^2 = \frac{1}{2} \sum_n \|f(\mathbf{x}_n, \mathbf{W}) - y_n\|_2^2$$

where \mathbf{x}_n is the input of the n^{th} example

y_n is the label of the n^{th} example

$f(\mathbf{x}_n, \mathbf{W})$ is the output of the neural net

Sequential Gradient Descent

- For each example (\mathbf{x}_n, y_n) adjust the weights as follows:

$$w_{ji} \leftarrow w_{ji} - \eta \frac{\partial E_n}{\partial w_{ji}}$$

- How can we compute the gradient efficiently given an arbitrary network structure?
- Answer: **backpropagation algorithm**
- Today: **automatic differentiation**

Backpropagation Algorithm

- Two phases:
 - Forward phase: compute output z_j of each unit j
 - Backward phase: compute delta δ_j at each unit j

Forward phase

- Propagate inputs forward to compute the output of each unit
- Output z_j at unit j :

$$z_j = h(a_j) \quad \text{where} \quad a_j = \sum_i w_{ji} z_i$$

Backward phase

- Use chain rule to recursively compute gradient

- For each weight w_{ji} : $\frac{\partial E_n}{\partial w_{ji}} = \frac{\partial E_n}{\partial a_j} \frac{\partial a_j}{\partial w_{ji}} = \delta_j z_i$

- Let $\delta_j \equiv \frac{\partial E_n}{\partial a_j}$ then

$$\delta_j = \begin{cases} h'(a_j)(z_j - y_j) & \text{base case: } j \text{ is an output unit} \\ h'(a_j) \sum_k w_{kj} \delta_k & \text{recursion: } j \text{ is a hidden unit} \end{cases}$$

- Since $a_j = \sum_i w_{ji} z_i$ then $\frac{\partial a_j}{\partial w_{ji}} = z_i$

Simple Example

- Consider a network with two layers:
 - Hidden nodes: $h(a) = \tanh(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}}$
 - Tip: $\tanh'(a) = 1 - (\tanh(a))^2$
 - Output node: $h(a) = a$
- Objective: squared error

Simple Example

- Forward propagation:

- Hidden units: $a_j =$

$$z_j =$$

- Output units: $a_k =$

$$z_k =$$

- Backward propagation:

- Output units: $\delta_k =$

- Hidden units: $\delta_j =$

- Gradients:

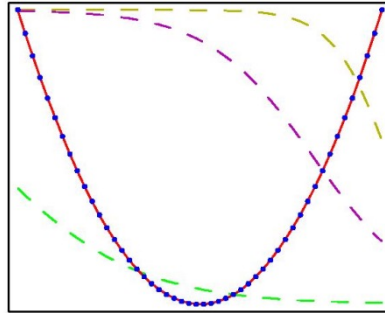
- Hidden layers: $\frac{\partial E_n}{\partial w_{ji}} =$

- Output layer: $\frac{\partial E_n}{\partial w_{kj}} =$

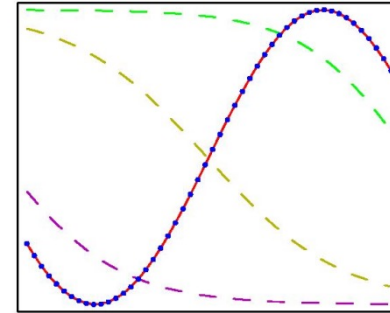
Non-linear regression examples

- Two-layer network:
 - 3 tanh hidden units and 1 identity output unit

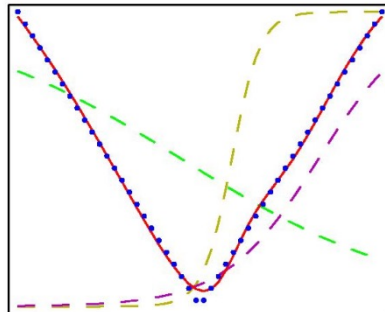
$$y = x^2$$



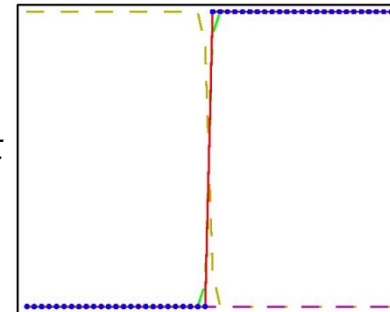
$$y = \sin x$$



$$y = |x|$$



$$y = \int_{-\infty}^x \delta(t) dt$$



Analysis

- Efficiency:
 - Fast gradient computation: linear in number of weights
- Convergence:
 - Slow convergence (linear rate)
 - May get trapped in local optima
- Prone to overfitting
 - Solutions: early stopping, regularization (add $\|w\|_2^2$ penalty term to objective), dropout